1

MAXWELL'S INTEGRAL LAWS IN FREE SPACE

1.0 INTRODUCTION

Practical, intellectual, and cultural reasons motivate the study of electricity and magnetism. The operation of electrical systems designed to perform certain engineering tasks depends, at least in part, on electrical, electromechanical, or electrochemical phenomena. The electrical aspects of these applications are described by Maxwell's equations. As a description of the temporal evolution of electromagnetic fields in three-dimensional space, these same equations form a concise summary of a wider range of phenomena than can be found in any other discipline. Maxwell's equations are an intellectual achievement that should be familiar to every student of physical phenomena. As part of the theory of fields that includes continuum mechanics, quantum mechanics, heat and mass transfer, and many other disciplines, our subject develops the mathematical language and methods that are the basis for these other areas.

For those who have an interest in electromechanical energy conversion, transmission systems at power or radio frequencies, waveguides at microwave or optical frequencies, antennas, or plasmas, there is little need to argue the necessity for becoming expert in dealing with electromagnetic fields. There are others who may require encouragement. For example, circuit designers may be satisfied with circuit theory, the laws of which are stated in terms of voltages and currents and in terms of the relations imposed upon the voltages and currents by the circuit elements. However, these laws break down at high frequencies, and this cannot be understood without electromagnetic field theory. The limitations of circuit models come into play as the frequency is raised so high that the propagation time of electromagnetic fields becomes comparable to a period, with the result that "inductors" behave as "capacitors" and vice versa. Other limitations are associated with loss phenomena. As the frequency is raised, resistors and transistors are limited by "capacitive" effects, and transducers and transformers by "eddy" currents.

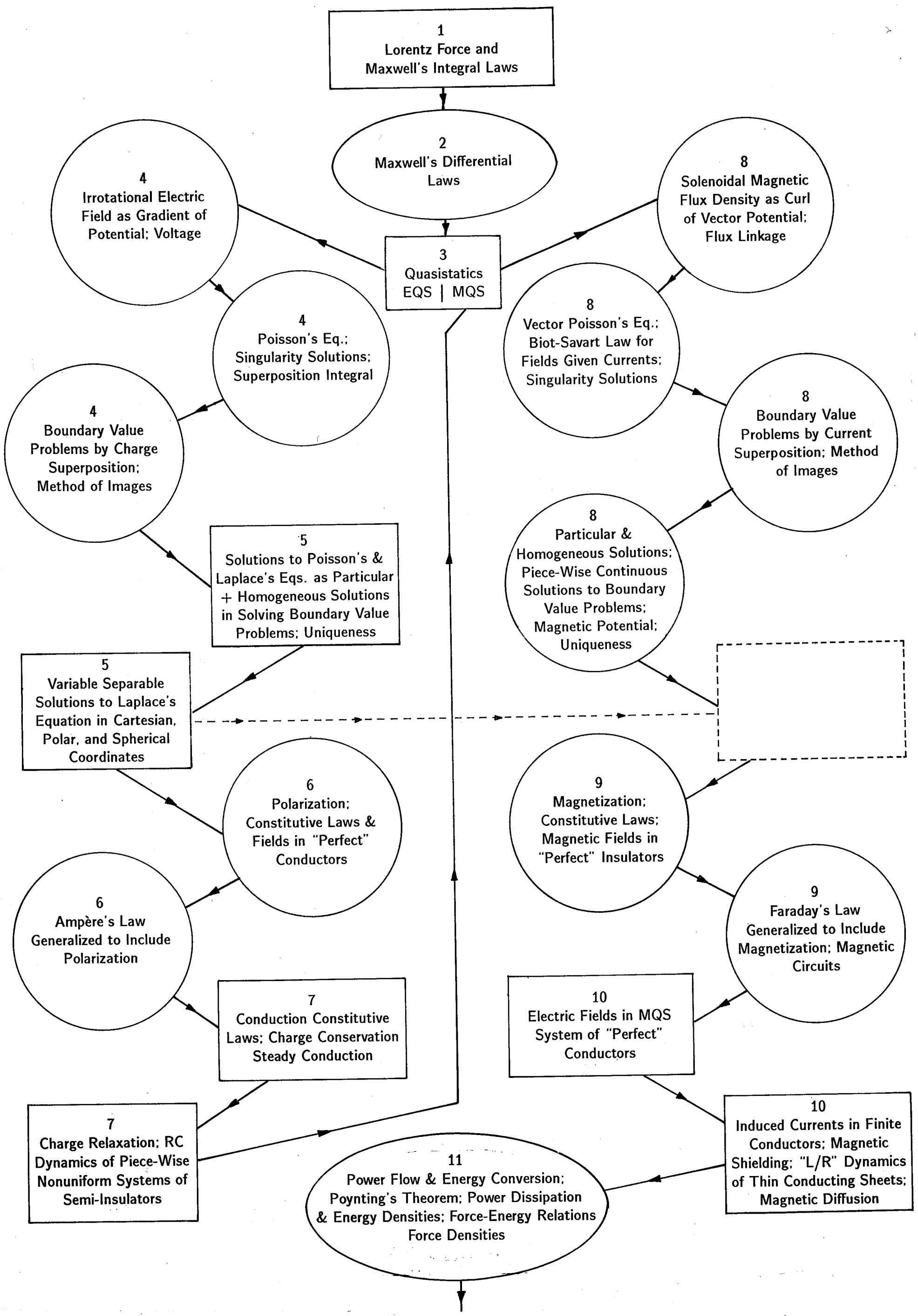
Anyone concerned with developing circuit models for physical systems requires a field theory background to justify approximations and to derive the values of the circuit parameters. Thus, the bioengineer concerned with electrocardiography or neurophysiology must resort to field theory in establishing a meaningful connection between the physical reality and models, when these are stated in terms of circuit elements. Similarly, even if a control theorist makes use of a lumped parameter model, its justification hinges on a continuum theory, whether electromagnetic, mechanical, or thermal in nature.

Computer hardware may seem to be another application not dependent on electromagnetic field theory. The software interface through which the computer is often seen makes it seem unrelated to our subject. Although the hardware is generally represented in terms of circuits, the practical realization of a computer designed to carry out logic operations is limited by electromagnetic laws. For example, the signal originating at one point in a computer cannot reach another point within a time less than that required for a signal, propagating at the speed of light, to traverse the interconnecting wires. That circuit models have remained useful as computation speeds have increased is a tribute to the solid state technology that has made it possible to decrease the size of the fundamental circuit elements. Sooner or later, the fundamental limitations imposed by the electromagnetic fields define the computation speed frontier of computer technology, whether it be caused by electromagnetic wave delays or electrical power dissipation.

Overview of Subject. As illustrated diagrammatically in Fig. 1.0.1, we start with Maxwell's equations written in integral form. This chapter begins with a definition of the fields in terms of forces and sources followed by a review of each of the integral laws. Interwoven with the development are examples intended to develop the methods for surface and volume integrals used in stating the laws. The examples are also intended to attach at least one physical situation to each of the laws. Our objective in the chapters that follow is to make these laws useful, not only in modeling engineering systems but in dealing with practical systems in a qualitative fashion (as an inventor often does). The integral laws are directly useful for (a) dealing with fields in this qualitative way, (b) finding fields in simple configurations having a great deal of symmetry, and (c) relating fields to their sources.

Chapter 2 develops a differential description from the integral laws. By following the examples and some of the homework associated with each of the sections, a minimum background in the mathematical theorems and operators is developed. The differential operators and associated integral theorems are brought in as needed. Thus, the divergence and curl operators, along with the theorems of Gauss and Stokes, are developed in Chap. 2, while the gradient operator and integral theorem are naturally derived in Chap. 4.

Static fields are often the first topic in developing an understanding of phenomena predicted by Maxwell's equations. Fields are not measurable, let alone of practical interest, unless they are dynamic. As developed here, fields are never truly static. The subject of quasistatics, begun in Chap. 3, is central to the approach we will use to understand the implications of Maxwell's equations. A mature understanding of these equations is achieved when one has learned how to neglect complications that are inconsequential. The electroquasistatic (EQS) and magne-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Lorentz Force and
Maxwell's Integral Laws
2
Maxwell's Differential
Laws
3
Quasistatics
EQS | MQS
4
Irrotational Electric
Field as Gradient of
Potential; Voltage
4
Poisson's Eq.;
Singularity Solutions;
Superposition Integral
4
Boundary Value
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Superposition;
Method of Images
5
Solutions to Poisson's &
Laplace's Eqs. as Particular
+ Homogeneous Solutions
in Solving Boundary Value
Problems; Uniqueness
5
Variable Separable
Solutions to Laplace's
Equation in Cartesian,
Polar, and Spherical
Coordinates
6
Polarization;
Constitutive Laws &
Fields in "Perfect"
Conductors
6
Ampère's Law
Generalized to Include
Polarization
7
Conduction Constitutive
Laws; Charge Conservation
Steady Conduction
7
Charge Relaxation; RC
Dynamics of Piece-Wise
Nonuniform Systems of
Semi-Insulators
8
Solenoidal Magnetic
Flux Density as Curl
of Vector Potential;
Flux Linkage
8
Vector Poisson's Eq.;
Biot-Savart Law for
Fields Given Currents;
Singularity Solutions
8
Boundary Value
Problems by Current
Superposition; Method
of Images
8
Particular &
Homogeneous Solutions;
Piece-Wise Continuous
Solutions to Boundary
Value Problems;
Magnetic Potential;
Uniqueness
9
Magnetization;
Constitutive Laws;
Magnetic Fields in
"Perfect" Insulators
9
Faraday's Law
Generalized to Include
Magnetization; Magnetic
Circuits
10
Electric Fields in MQS
System of "Perfect"
Conductors
10
Induced Currents in Finite
Conductors; Magnetic
Shielding; "L/R" Dynamics
of Thin Conducting Sheets;
Magnetic Diffusion
11
Power Flow & Energy Conversion;
Poynting's Theorem; Power Dissipation
& Energy Densities; Force-Energy Relations
Force Densities

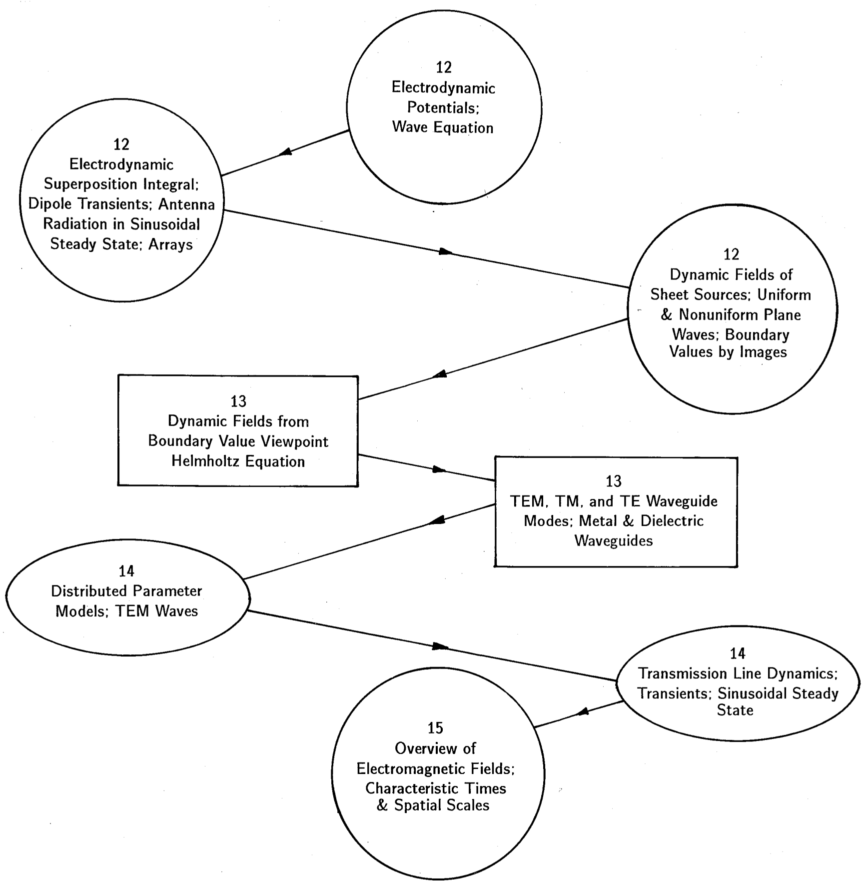

Fig. 1.0.1 Outline of Subject. The three columns, respectively for electroquasistatics, magnetoquasistatics and electrodynamics, show parallels in development.

toquasistatic (MQS) approximations are justified if time rates of change are slow enough (frequencies are low enough) so that time delays due to the propagation of electromagnetic waves are unimportant. The examples considered in Chap. 3 give some notion as to which of the two approximations is appropriate in a given situation. A full appreciation for the quasistatic approximations will come into view as the EQS and MQS developments are drawn together in Chaps. 11 through 15.

Although capacitors and inductors are examples in the electroquasistatic and magnetoquasistatic categories, respectively, it is not true that quasistatic systems can be generally modeled by frequency-independent circuit elements. High-frequency models for transistors are correctly based on the EQS approximation. Electromagnetic wave delays in the transistors are not consequential. Nevertheless, dynamic effects are important and the EQS approximation can contain the finite time for charge migration. Models for eddy current shields or heaters are correctly based on the MQS approximation. Again, the delay time of an electromagnetic wave is unimportant while the all-important diffusion time of the magnetic field is represented by the MQS laws. Space charge waves on an electron beam or spin waves in a saturated magnetizable material are often described by EQS and MQS laws, respectively, even though frequencies of interest are in the GHz range.

The parallel developments of EQS (Chaps. 4–7) and MQS systems (Chaps. 8–10) is emphasized by the first page of Fig. 1.0.1. For each topic in the EQS column to the left there is an analogous one at the same level in the MQS column. Although the field concepts and mathematical techniques used in dealing with EQS and MQS systems are often similar, a comparative study reveals as many contrasts as direct analogies. There is a two-way interplay between the electric and magnetic studies. Not only are results from the EQS developments applied in the description of MQS systems, but the examination of MQS situations leads to a greater appreciation for the EQS laws.

At the tops of the EQS and the MQS columns, the first page of Fig. 1.0.1, general (contrasting) attributes of the electric and magnetic fields are identified. The developments then lead from situations where the field sources are prescribed to where they are to be determined. Thus, EQS electric fields are first found from prescribed distributions of charge, while MQS magnetic fields are determined given the currents. The development of the EQS field solution is a direct investment in the subsequent MQS derivation. It is then recognized that in many practical situations, these sources are induced in materials and must therefore be found as part of the field solution. In the first of these situations, induced sources are on the boundaries of conductors having a sufficiently high electrical conductivity to be modeled as "perfectly" conducting. For the EQS systems, these sources are surface charges, while for the MQS, they are surface currents. In either case, fields must satisfy boundary conditions, and the EQS study provides not only mathematical techniques but even partial differential equations directly applicable to MQS problems.

Polarization and magnetization account for field sources that can be prescribed (electrets and permanent magnets) or induced by the fields themselves. In the Chu formulation used here, there is a complete analogy between the way in which polarization and magnetization are represented. Thus, there is a direct transfer of ideas from Chap. 6 to Chap. 9.

The parallel quasistatic studies culminate in Chaps. 7 and 10 in an examination of loss phenomena. Here we learn that very different answers must be given to the question "When is a conductor perfect?" for EQS on one hand, and MQS on the other.

In Chap. 11, many of the concepts developed previously are put to work through the consideration of the flow of power, storage of energy, and production of electromagnetic forces. From this chapter on, Maxwell's equations are used without approximation. Thus, the EQS and MQS approximations are seen to represent systems in which either the electric or the magnetic energy storage dominates respectively.

In Chaps. 12 through 14, the focus is on electromagnetic waves. The development is a natural extension of the approach taken in the EQS and MQS columns. This is emphasized by the outline represented on the right page of Fig. 1.0.1. The topics of Chaps. 12 and 13 parallel those of the EQS and MQS columns on the previous page. Potentials used to represent electrodynamic fields are a natural generalization of those used for the EQS and MQS systems. As for the quasistatic fields, the fields of given sources are considered first. An immediate practical application is therefore the description of radiation fields of antennas.

The boundary value point of view, introduced for EQS systems in Chap. 5 and for MQS systems in Chap. 8, is the basic theme of Chap. 13. Practical examples include simple transmission lines and waveguides. An understanding of transmission line dynamics, the subject of Chap. 14, is necessary in dealing with the "conventional" ideal lines that model most high-frequency systems. They are also shown to provide useful models for representing quasistatic dynamical processes.

To make practical use of Maxwell's equations, it is necessary to master the art of making approximations. Based on the electromagnetic properties and dimensions of a system and on the time scales (frequencies) of importance, how can a physical system be broken into electromagnetic subsystems, each described by its dominant physical processes? It is with this goal in mind that the EQS and MQS approximations are introduced in Chap. 3, and to this end that Chap. 15 gives an overview of electromagnetic fields.

1.1 THE LORENTZ LAW IN FREE SPACE

There are two points of view for formulating a theory of electrodynamics. The older one views the forces of attraction or repulsion between two charges or currents as the result of action at a distance. Coulomb's law of electrostatics and the corresponding law of magnetostatics were first stated in this fashion. Faraday[1] introduced a new approach in which he envisioned the space between interacting charges to be filled with fields, by which the space is activated in a certain sense; forces between two interacting charges are then transferred, in Faraday's view, from volume element to volume element in the space between the interacting bodies until finally they are transferred from one charge to the other. The advantage of Faraday's approach was that it brought to bear on the electromagnetic problem the then well-developed theory of continuum mechanics. The culmination of this point of view was Maxwell's formulation[2] of the equations named after him.

From Faraday's point of view, electric and magnetic fields are defined at a point $\mathbf{r}$ even when there is no charge present there. The fields are defined in terms of the force that would be exerted on a test charge q if it were introduced at $\mathbf{r}$ moving at a velocity $\mathbf{v}$ at the time of interest. It is found experimentally that such a force would be composed of two parts, one that is independent of $\mathbf{v}$, and the other proportional to $\mathbf{v}$ and orthogonal to it. The force is summarized in terms of the *electric field intensity* $\mathbf{E}$ and *magnetic flux density* $\mu_o\mathbf{H}$ by the *Lorentz force law.* (For a review of vector operations, see Appendix 1.)

$$\boxed{\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mu_o\mathbf{H})} \qquad (1)$$

The superposition of electric and magnetic force contributions to (1) is illustrated in Fig. 1.1.1. Included in the figure is a reminder of the right-hand rule used to determine the direction of the cross-product of $\mathbf{v}$ and $\mu_o\mathbf{H}$. In general, $\mathbf{E}$ and $\mathbf{H}$ are not uniform, but rather are functions of position $\mathbf{r}$ and time t: $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mu_o\mathbf{H} = \mu_o\mathbf{H}(\mathbf{r}, t)$.

In addition to the units of length, mass, and time associated with mechanics, a unit of charge is required by the theory of electrodynamics. This unit is the

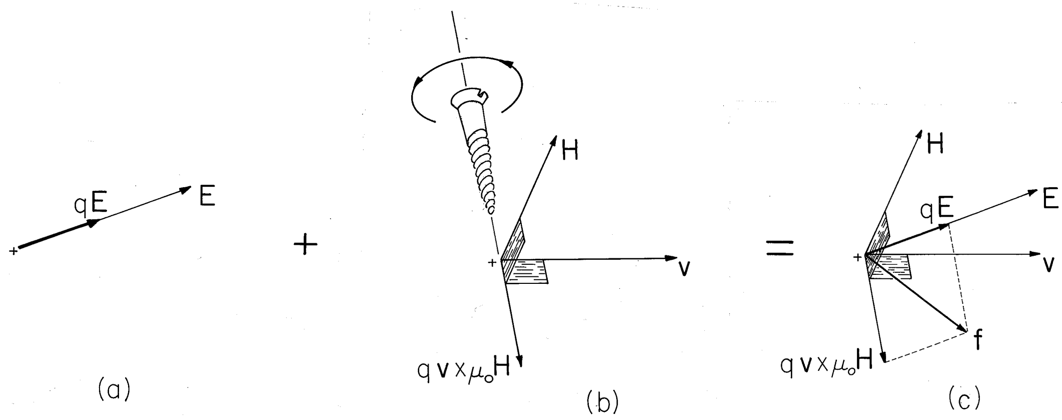

Fig. 1.1.1 Lorentz force **f** in geometric relation to the electric and magnetic field intensities, **E** and **H**, and the charge velocity **v**: (a) electric force, (b) magnetic force, and (c) total force.

coulomb. The Lorentz force law, (1), then serves to define the units of **E** and of $\mu_o\mathbf{H}$.

$$\text{units of } \mathbf{E} = \frac{\text{newton}}{\text{coulomb}} = \frac{\text{kilogram meter}/(\text{second})^2}{\text{coulomb}} \tag{2}$$

$$\text{units of } \mu_o\mathbf{H} = \frac{\text{newton}}{\text{coulomb meter}/\text{second}} = \frac{\text{kilogram}}{\text{coulomb second}} \tag{3}$$

We can only establish the units of the magnetic flux density $\mu_o\mathbf{H}$ from the force law and cannot argue until Sec. 1.4 that the derived units of **H** are ampere/meter and hence of μ_o are henry/meter.

In much of electrodynamics, the predominant concern is not with mechanics but with electric and magnetic fields in their own right. Therefore, it is inconvenient to use the unit of mass when checking the units of quantities. It proves useful to introduce a new name for the unit of electric field intensity– the unit of volt/meter.

In the summary of variables given in Table 1.8.2 at the end of the chapter, the fundamental units are SI, while the derived units exploit the fact that the unit of mass, kilogram = volt-coulomb-second^2/meter^2 and also that a coulomb/second = ampere. Dimensional checking of equations is guaranteed if the basic units are used, but may often be accomplished using the derived units. The latter communicate the physical nature of the variable and the natural symmetry of the electric and magnetic variables.

Example 1.1.1. Electron Motion in Vacuum in a Uniform Static Electric Field

In vacuum, the motion of a charged particle is limited only by its own inertia. In the uniform electric field illustrated in Fig. 1.1.2, there is no magnetic field, and an electron starts out from the plane $x = 0$ with an initial velocity v_i.
The "imposed" electric field is $\mathbf{E} = \mathbf{i_x}E_x$, where $\mathbf{i_x}$ is the unit vector in the x direction and E_x is a given constant. The trajectory is to be determined here and used to exemplify the charge and current density in Example 1.2.1.

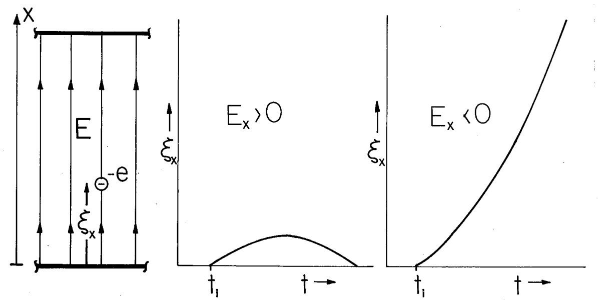

Fig. 1.1.2 An electron, subject to the uniform electric field intensity $\mathbf{E}_x$, has the position ξ_x, shown as a function of time for positive and negative fields.

With m defined as the electron mass, Newton's law combines with the Lorentz law to describe the motion.

$$m\frac{d^2\xi_x}{dt^2} = f = -eE_x \tag{4}$$

The electron position ξ_x is shown in Fig. 1.1.2. The charge of the electron is customarily denoted by e ($e = 1.6 \times 10^{-19}$ coulomb) where e is positive, thus necessitating an explicit minus sign in (4).

By integrating twice, we get

$$\xi_x = -\frac{1}{2}\frac{e}{m}E_x t^2 + c_1 t + c_2 \tag{5}$$

where c_1 and c_2 are integration constants. If we assume that the electron is at $\xi_x = 0$ and has velocity v_i when $t = t_i$, it follows that these constants are

$$c_1 = v_i + \frac{e}{m}E_x t_i; \qquad c_2 = -v_i t_i - \frac{1}{2}\frac{e}{m}E_x t_i^2 \tag{6}$$

Thus, the electron position and velocity are given as a function of time by

$$\xi_x = -\frac{1}{2}\frac{e}{m}E_x(t - t_i)^2 + v_i(t - t_i) \tag{7}$$

$$\frac{d\xi_x}{dt} = -\frac{e}{m}E_x(t - t_i) + v_i \tag{8}$$

With x defined as upward and $E_x > 0$, the motion of an electron in an electric field is analogous to the free fall of a mass in a gravitational field, as illustrated by Fig. 1.1.2. With $E_x < 0$, and the initial velocity also positive, the velocity is a monotonically increasing function of time, as also illustrated by Fig. 1.1.2.

Example 1.1.2. Electron Motion in Vacuum in a Uniform Static Magnetic Field

The magnetic contribution to the Lorentz force is perpendicular to both the particle velocity and the imposed field. We illustrate this fact by considering the trajectory

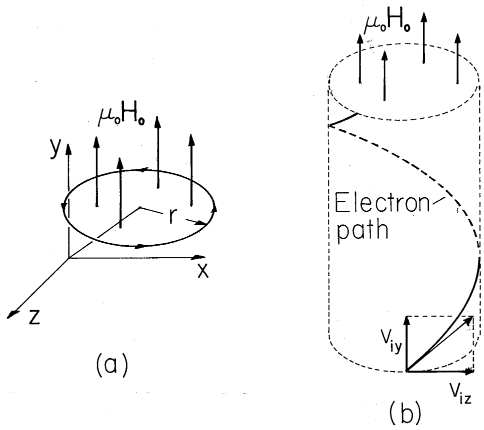

Fig. 1.1.3 (a) In a uniform magnetic flux density $\mu_o \mathbf{H}_o$ and with no initial velocity in the y direction, an electron has a circular orbit. (b) With an initial velocity in the y direction, the orbit is helical.

resulting from an initial velocity v_{iz} along the z axis. With a uniform constant magnetic flux density $\mu_o \mathbf{H}$ existing along the y axis, the force is

$$\mathbf{f} = -e(\mathbf{v} \times \mu_o \mathbf{H}) \tag{9}$$

The cross-product of two vectors is perpendicular to the two vector factors, so the acceleration of the electron, caused by the magnetic field, is always perpendicular to its velocity. Therefore, a magnetic field alone cannot change the magnitude of the electron velocity (and hence the kinetic energy of the electron) but can change only the direction of the velocity. Because the magnetic field is uniform, because the velocity and the rate of change of the velocity lie in a plane perpendicular to the magnetic field, and, finally, because the magnitude of $\mathbf{v}$ does not change, we find that the acceleration has a constant magnitude and is orthogonal to both the velocity and the magnetic field. The electron moves in a circle so that the centrifugal force counterbalances the magnetic force. Figure 1.1.3a illustrates the motion. The radius of the circle is determined by equating the centrifugal force and radial Lorentz force

$$e\mu_o |v| H_o = \frac{mv^2}{r} \tag{10}$$

which leads to

$$r = \frac{m}{e} \frac{|v|}{\mu_o H_o} \tag{11}$$

The foregoing problem can be modified to account for any arbitrary initial angle between the velocity and the magnetic field. The vector equation of motion (really three equations in the three unknowns ξ_x, ξ_y, ξ_z)

$$m \frac{d^2 \bar{\xi}}{dt^2} = -e\Big(\frac{d\bar{\xi}}{dt} \times \mu_o \mathbf{H}\Big) \tag{12}$$

is linear in $\bar{\xi}$, and so solutions can be superimposed to satisfy initial conditions that include not only a velocity v_{iz} but one in the y direction as well, v_{iy}. Motion in the same direction as the magnetic field does not give rise to an additional force. Thus, the y component of (12) is zero on the right. An integration then shows that the y directed velocity remains constant at its initial value, v_{iy}. This uniform motion can be added to that already obtained to see that the electron follows a helical path, as shown in Fig. 1.1.3b.

It is interesting to note that the angular frequency of rotation of the electron around the field is independent of the speed of the electron and depends only upon the magnetic flux density, $\mu_o H_o$. Indeed, from (11) we find

$$\frac{v}{r} \equiv \omega_c = \frac{e}{m}\mu_o H_o \tag{13}$$

For a flux density of 1 volt-second/meter (or 1 tesla), the *cyclotron frequency* is $f_c = \omega_c/2\pi = 28\,\text{GHz}$. (For an electron, $e = 1.602\times10^{-19}$ coulomb and $m = 9.106\times10^{-31}$ kg.) With an initial velocity in the z direction of 3×10^7 m/s, the radius of gyration in the flux density $\mu_o H = 1$ tesla is $r = v_{iz}/\omega_c = 1.7\times10^{-4}$ m.

1.2 CHARGE AND CURRENT DENSITIES

In Maxwell's day, it was not known that charges are not infinitely divisible but occur in elementary units of 1.6×10^{-19} coulomb, the charge of an electron. Hence, Maxwell's macroscopic theory deals with continuous charge distributions. This is an adequate description for fields of engineering interest that are produced by aggregates of large numbers of elementary charges. These aggregates produce charge distributions that are described conveniently in terms of a charge per unit volume, a charge density ρ.

Pick an incremental volume and determine the net charge within. Then

$$\rho(\mathbf{r}, t) \equiv \frac{\text{net charge in } \Delta\text{V}}{\Delta V} \tag{1}$$

is the *charge density* at the position $\mathbf{r}$ when the time is t. The units of ρ are coulomb/meter^3. The volume ΔV is chosen small as compared to the dimensions of the system of interest, but large enough so as to contain many elementary charges. The charge density ρ is treated as a continuous function of position. The "graininess" of the charge distribution is ignored in such a "macroscopic" treatment.

Fundamentally, current is charge transport and connotes the time rate of change of charge. Current density is a directed current per unit area and hence measured in (coulomb/second)/meter^2. A charge density ρ moving at a velocity $\mathbf{v}$ implies a rate of charge transport per unit area, a *current density* $\mathbf{J}$, given by

$$\mathbf{J} = \rho\mathbf{v} \tag{2}$$

One way to envision this relation is shown in Fig. 1.2.1, where a charge density ρ having velocity $\mathbf{v}$ traverses a differential area δa. The area element has a unit normal $\mathbf{n}$, so that a differential area vector can be defined as $\delta\mathbf{a} = \mathbf{n}\delta a$. The charge that passes during a differential time δt is equal to the total charge contained in the volume $\mathbf{v}\cdot\delta\mathbf{a}dt$. Therefore,

$$d(\delta q) = \rho\mathbf{v}\cdot\delta\mathbf{a}dt \tag{3}$$

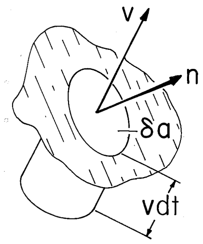

Fig. 1.2.1 Current density **J** passing through surface having a normal **n**.

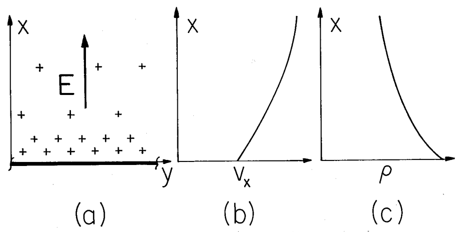

Fig. 1.2.2 Charge injected at the lower boundary is accelerated upward by an electric field. Vertical distributions of (a) field intensity, (b) velocity and (c) charge density.

Divided by dt, we expect (3) to take the form $\mathbf{J} \cdot \delta \mathbf{a}$, so it follows that the current density is related to the charge density by (2).

The velocity $\mathbf{v}$ is the velocity of the charge. Just how the charge is set into motion depends on the physical situation. The charge might be suspended in or on an insulating material which is itself in motion. In that case, the velocity would also be that of the material. More likely, it is the result of applying an electric field to a conductor, as considered in Chap. 7. For charged particles moving in vacuum, it might result from motions represented by the laws of Newton and Lorentz, as illustrated in the examples in Sec.1.1. This is the case in the following example.

Example 1.2.1. Charge and Current Densities in a Vacuum Diode

Consider the charge and current densities for electrons being emitted with initial velocity $\mathbf{v}$ from a "cathode" in the plane $x = 0$, as shown in Fig. 1.2.2a.[1]

Electrons are continuously injected. As in Example 1.1.1, where the motions of the individual electrons are considered, the electric field is assumed to be uniform. In the next section, it is recognized that charge is the source of the electric field. Here it is assumed that the charge used to impose the uniform field is much greater than the "space charge" associated with the electrons. This is justified in the limit of a low electron current. Any one of the electrons has a position and velocity given by (1.1.7) and (1.1.8). If each is injected with the same initial velocity, the charge and current densities in any given plane $x =$ constant would be expected to be independent of time. Moreover, the current passing any x-plane should be the same as that passing any other such plane. That is, in the steady state, the current density is independent

[1] Here we picture the field variables E_x, v_x, and ρ as though they were positive. For electrons, $\rho < 0$, and to make $v_x > 0$, we must have $E_x < 0$.

of not only time but x as well. Thus, it is possible to write

$$\rho(x)v_x(x) = J_o \tag{4}$$

where J_o is a given current density.

The following steps illustrate how this condition of current continuity makes it possible to shift from a description of the particle motions described with time as the independent variable to one in which coordinates (x, y, z) (or for short $\mathbf{r}$) are the independent coordinates. The relation between time and position for the electron described by (1.1.7) takes the form of a quadratic in $(t - t_i)$

$$\frac{1}{2}\frac{e}{m}E_x(t - t_i)^2 - v_i(t - t_i) + \xi_x = 0 \tag{5}$$

This can be solved to give the elapsed time for a particle to reach the position ξ_x. Note that of the two possible solutions to (5), the one selected satisfies the condition that when $t = t_i$, $\xi_x = 0$.

$$t - t_i = \frac{v_i - \sqrt{v_i^2 - 2\frac{e}{m}E_x\xi_x}}{\frac{e}{m}E_x} \tag{6}$$

With the benefit of this expression, the velocity given by (1.1.8) is written as

$$\frac{d\xi_x}{dt} = \sqrt{v_i^2 - \frac{2e}{m}E_x\xi_x} \tag{7}$$

Now we make a shift in viewpoint. On the left in (7) is the velocity v_x of the particle that is at the location $\xi_x = x$. Substitution of variables then gives

$$v_x = \sqrt{v_i^2 - 2\frac{e}{m}E_x x} \tag{8}$$

so that x becomes the independent variable used to express the dependent variable v_x. It follows from this expression and (4) that the charge density

$$\rho = \frac{J_o}{v_x} = \frac{J_o}{\sqrt{v_i^2 - \frac{2e}{m}E_x x}} \tag{9}$$

is also expressed as a function of x. In the plots shown in Fig. 1.2.2, it is assumed that $E_x < 0$, so that the electrons have velocities that increase monotonically with x. As should be expected, the charge density decreases with x because as they speed up, the electrons thin out to keep the current density constant.

1.3 GAUSS' INTEGRAL LAW OF ELECTRIC FIELD INTENSITY

The Lorentz force law of Sec. 1.1 expresses the effect of electromagnetic fields on a moving charge. The remaining sections in this chapter are concerned with the reaction of the moving charges upon the electromagnetic fields. The first of

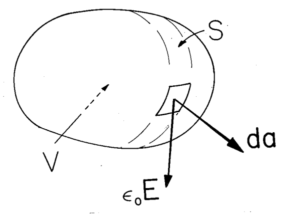

Fig. 1.3.1 General surface S enclosing volume V.

Maxwell's equations to be considered, *Gauss' law*, describes how the electric field intensity is related to its source. The net charge within an arbitrary volume V that is enclosed by a surface S is related to the net electric flux through that surface by

$$\oint_S \epsilon_o \mathbf{E} \cdot d\mathbf{a} = \int_V \rho dv \tag{1}$$

With the surface normal defined as directed outward, the volume is shown in Fig. 1.3.1. Here the *permittivity of free space*, $\epsilon_o = 8.854 \times 10^{-12}$ farad/meter, is an empirical constant needed to express Maxwell's equations in SI units. On the right in (1) is the net charge enclosed by the surface S. On the left is the summation over this same closed surface of the differential contributions of flux $\epsilon_o \mathbf{E} \cdot d\mathbf{a}$. The quantity $\epsilon_o \mathbf{E}$ is called the *electric displacement flux density* and, [from (1)], has the units of coulomb/meter^2. Out of any region containing net charge, there must be a net displacement flux.

The following example illustrates the mechanics of carrying out the volume and surface integrations.

Example 1.3.1. Electric Field Due to Spherically Symmetric Charge Distribution

Given the charge and current distributions, the integral laws fully determine the electric and magnetic fields. However, they are not directly useful unless there is a great deal of symmetry. An example is the distribution of charge density

$$\rho(r) = \begin{cases} \rho_o \frac{r}{R}; & r < R \\ 0; & r > R \end{cases} \tag{2}$$

in the spherical coordinate system of Fig. 1.3.2. Here ρ_o and R are given constants. An argument based on the spherical symmetry shows that the only possible component of $\mathbf{E}$ is radial.

$$\mathbf{E} = \mathbf{i_r} E_r(r) \tag{3}$$

Indeed, suppose that in addition to this r component the field possesses a ϕ component. At a given point, the components of $\mathbf{E}$ then appear as shown in Fig. 1.3.2b. Rotation of the system about the axis shown results in a component of $\mathbf{E}$ in some new direction perpendicular to r. However, the rotation leaves the source of that field, the charge distribution, unaltered. It follows that E_ϕ must be zero. A similar argument shows that E_θ also is zero.

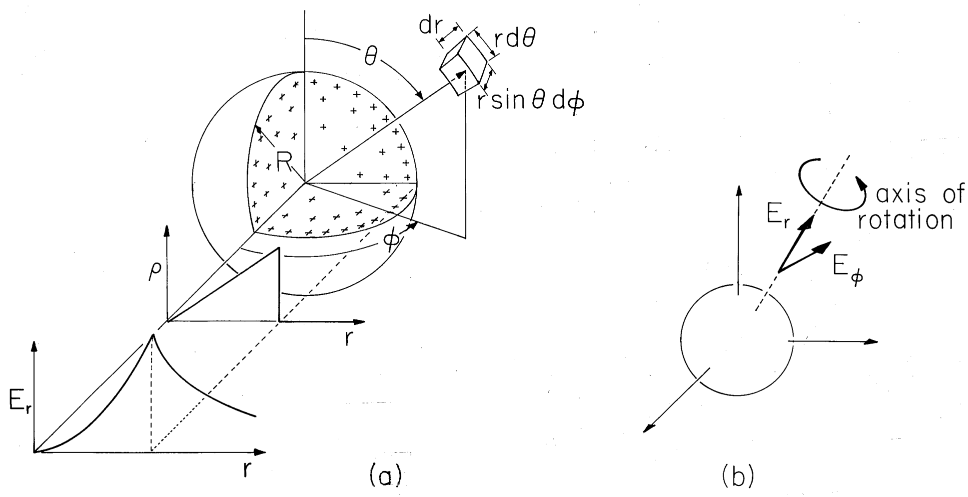

Fig. 1.3.2 (a) Spherically symmetric charge distribution, showing radial dependence of charge density and associated radial electric field intensity. (b) Axis of rotation for demonstration that the components of **E** transverse to the radial coordinate are zero.

The incremental volume element is

$$dv = (dr)(rd\theta)(r\sin\theta d\phi) \tag{4}$$

and it follows that for a spherical volume having arbitrary radius r,

$$\int_V \rho dv = \begin{cases} \int_0^r \int_0^\pi \int_0^{2\pi} \left[\rho_o \frac{r'}{R}\right](r'\sin\theta d\phi)(r'd\theta)dr' = \frac{\pi\rho_o}{R}r^4; & r < R \\ \int_0^R \int_0^\pi \int_0^{2\pi} \left[\rho_o \frac{r'}{R}\right](r'\sin\theta d\phi)(r'd\theta)dr' = \pi\rho_o R^3; & R < r \end{cases} \tag{5}$$

To evaluate the left-hand side of (1), note that

$$\mathbf{n} = \mathbf{i_r}; \qquad d\mathbf{a} = \mathbf{i_r}(rd\theta)(r\sin\theta d\phi) \tag{6}$$

Thus, for the spherical surface at the arbitrary radius r,

$$\oint_S \epsilon_o \mathbf{E}\cdot d\mathbf{a} = \int_0^\pi \int_0^{2\pi} \epsilon_o E_r (r\sin\theta d\phi)(rd\theta) = \epsilon_o E_r 4\pi r^2 \tag{7}$$

With the volume and surface integrals evaluated in (5) and (7), Gauss' law, (1), shows that

$$\epsilon_o E_r 4\pi r^2 = \frac{\pi\rho_o}{R}r^4 \Rightarrow E_r = \frac{\rho_o r^2}{4\epsilon_o R}; \qquad r < R \tag{8a}$$

$$\epsilon_o E_r 4\pi r^2 = \pi\rho_o R^3 \Rightarrow E_r = \frac{\rho_o R^3}{4\epsilon_o r^2}; \qquad R < r \tag{8b}$$

Inside the spherical charged region, the radial electric field increases with the square of the radius because even though the associated surface increases like the square

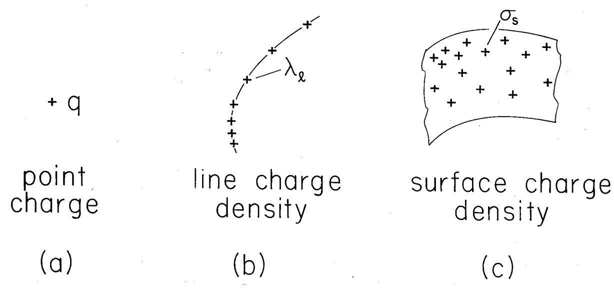

Fig. 1.3.3 Singular charge distributions: (a) point charge, (b) line charge, (c) surface charge.

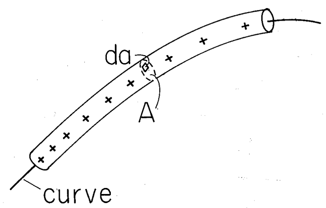

Fig. 1.3.4 Filamentary volume element having cross-section *da* used to define line charge density.

of the radius, the enclosed charge increases even more rapidly. Figure 1.3.2 illustrates this dependence, as well as the exterior field decay. Outside, the surface area continues to increase in proportion to r^2, but the enclosed charge remains constant.

Singular Charge Distributions. Examples of singular functions from circuit theory are impulse and step functions. Because there is only the one independent variable, namely time, circuit theory is concerned with only one "dimension." In three-dimensional field theory, there are three spatial analogues of the temporal impulse function. These are point, line, and surface distributions of ρ, as illustrated in Fig. 1.3.3. Like the temporal impulse function of circuit theory, these singular distributions are defined in terms of integrals.

A *point charge* is the limit of an infinite charge density occupying zero volume. With q defined as the net charge,

$$q = \lim_{\substack{\rho\to\infty \\ V\to 0}} \int_V \rho dv \tag{9}$$

the point charge can be pictured as a small charge-filled region, the outside of which is charge free. An example is given in Fig. 1.3.2 in the limit where the volume $4\pi R^3/3$ goes to zero, while $q = \pi\rho_o R^3$ remains finite.

A *line charge density* represents a two-dimensional singularity in charge density. It is the mathematical abstraction representing a thin charge filament. In terms of the filamentary volume shown in Fig. 1.3.4, the line charge per unit length λ_l (the line charge density) is defined as the limit where the cross-sectional area of the volume goes to zero, ρ goes to infinity, but the integral

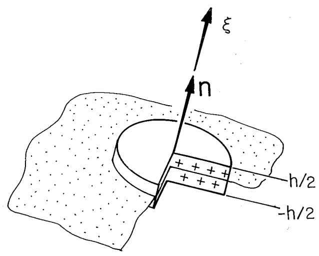

Fig. 1.3.5 Volume element having thickness h used to define surface charge density.

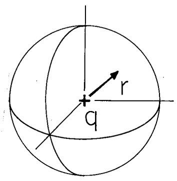

Fig. 1.3.6 Point charge q at origin of spherical coordinate system.

$$\lambda_l = \lim_{\substack{\rho\to\infty \\ A\to 0}} \int_A \rho da \tag{10}$$

remains finite. In general, λ_l is a function of position along the curve.

The one-dimensional singularity in charge density is represented by the *surface charge density.* The charge density is very large in the vicinity of a surface. Thus, as a function of a coordinate perpendicular to that surface, the charge density is a one-dimensional impulse function. To define the surface charge density, mount a pillbox as shown in Fig. 1.3.5 so that its top and bottom surfaces are on the two sides of the surface. The surface charge density is then defined as the limit

$$\sigma_s = \lim_{\substack{\rho\to\infty \\ h\to 0}} \int_{\xi-\frac{h}{2}}^{\xi+\frac{h}{2}} \rho d\xi \tag{11}$$

where the ξ coordinate is picked parallel to the direction of the normal to the surface, $\mathbf{n}$. In general, the surface charge density σ_s is a function of position in the surface.

Illustration. Field of a Point Charge

A point charge q is located at the origin in Fig. 1.3.6. There are no other charges. By the same arguments as used in Example 1.3.1, the spherical symmetry of the charge distribution requires that the electric field be radial and be independent of θ and ϕ. Evaluation of the surface integral in Gauss' integral law, (1), amounts to multiplying $\epsilon_o E_r$ by the surface area. Because all of the charge is concentrated at the origin, the volume integral gives q, regardless of radial position of the surface S. Thus,

$$4\pi r^2 \epsilon_o E_r = q \Rightarrow \mathbf{E} = \frac{q}{4\pi\epsilon_o r^2}\mathbf{i_r} \tag{12}$$

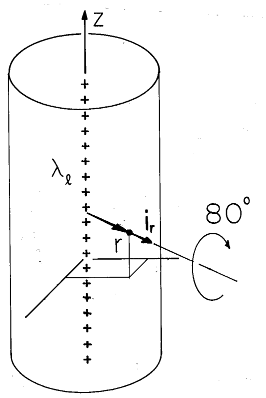

Fig. 1.3.7 Uniform line charge distributed from − infinity to + infinity along z axis. Rotation by 180 degrees about axis shown leads to conclusion that electric field is radial.

is the electric field associated with a point charge q.

Illustration. The Field Associated with Straight Uniform Line Charge

A uniform line charge is distributed along the z axis from $z = -\infty$ to $z = +\infty$, as shown in Fig. 1.3.7. For an observer at the radius r, translation of the line source in the z direction and rotation of the source about the z axis (in the ϕ direction) results in the same charge distribution, so the electric field must only depend on r. Moreover, $\mathbf{E}$ can only have a radial component. To see this, suppose that there were a z component of $\mathbf{E}$. Then a 180 degree rotation of the system about an axis perpendicular to and passing through the z axis must reverse this field. However, the rotation leaves the charge distribution unchanged. The contradiction is resolved only if $E_z = 0$. The same rotation makes it clear that E_ϕ must be zero.

This time, Gauss' integral law is applied using for S the surface of a right circular cylinder coaxial with the z axis and of arbitrary radius r. Contributions from the ends are zero because there the surface normal is perpendicular to $\mathbf{E}$. With the cylinder taken as having length l, the surface integration amounts to a multiplication of $\epsilon_o E_r$ by the surface area $2\pi rl$ while, the volume integral gives $l\lambda_l$ regardless of the radius r. Thus, (1) becomes

$$2\pi rl\epsilon_o E_r = \lambda_l l \Rightarrow \mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o r}\mathbf{i_r} \tag{13}$$

for the field of an infinitely long uniform line charge having density λ_l.

Example 1.3.2. The Field of a Pair of Equal and Opposite Infinite Planar Charge Densities

Consider the field produced by a surface charge density $+\sigma_o$ occupying all the $x-y$ plane at $z = s/2$ and an opposite surface charge density $-\sigma_o$ at $z = -s/2$.

First, the field must be z directed. Indeed there cannot be a component of $\mathbf{E}$ transverse to the z axis, because rotation of the system around the z axis leaves the same source distribution while rotating that component of $\mathbf{E}$. Hence, no such component exists.

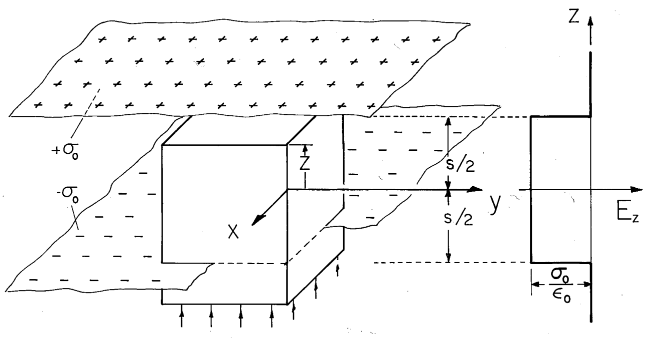

Fig. 1.3.8 Sheets of surface charge and volume of integration with upper surface at arbitrary position x. With field E_o due to external charges equal to zero, the distribution of electric field is the discontinuous function shown at right.

Because the source distribution is independent of x and y, E_z is independent of these coordinates. The z dependence is now established by means of Gauss' integral law, (1). The volume of integration, shown in Fig. 1.3.8, has cross-sectional area A in the $x - y$ plane. Its lower surface is located at an arbitrary fixed location below the lower surface charge distribution, while its upper surface is in the plane denoted by z. For now, we take E_z as being E_o on the lower surface. There is no contribution to the surface integral from the side walls because these have normals perpendicular to $\mathbf{E}$. It follows that Gauss' law, (1), becomes

$$\begin{aligned} A(\epsilon_o E_z - \epsilon_o E_o) &= 0; \qquad -\infty < z < -\frac{s}{2} \Rightarrow E_z = E_o \\ A(\epsilon_o E_z - \epsilon_o E_o) &= -A\sigma_o; \qquad -\frac{s}{2} < z < \frac{s}{2} \Rightarrow E_z = -\frac{\sigma_o}{\epsilon_o} + E_o \\ A(\epsilon_o E_z - \epsilon_o E_o) &= 0; \qquad \frac{s}{2} < z < \infty \Rightarrow E_z = E_o \end{aligned} \tag{14}$$

That is, with the upper surface below the lower charge sheet, no charge is enclosed by the surface of integration, and E_z is the constant E_o. With the upper surface of integration between the charge sheets, E_z is E_o minus σ_o/ϵ_o. Finally, with the upper integration surface above the upper charge sheet, E_z returns to its value of E_o. The external electric field E_o must be created by charges at $z = +\infty$, much as the field between the charge sheets is created by the given surface charges. Thus, if these charges at "infinity" are absent, $E_o = 0$, and the distribution of E_z is as shown to the right in Fig. 1.3.8.

Illustration. Coulomb's Force Law for Point Charges

It is worthwhile to see that for charges at rest, Gauss' integral law and the Lorentz force law give the familiar action at a distance force law. The force on a charge q is given by the Lorentz law, (1.1.1), and if the electric field is caused by a second charge at the origin in Fig. 1.3.9, then

$$\mathbf{f} = q\mathbf{E} = \frac{q_1 q_2}{4\pi\epsilon_o r^2}\mathbf{i_r} \tag{15}$$

Coulomb's famous statement that the force exerted by one charge on another is proportional to the product of their charges, acts along a line passing through each

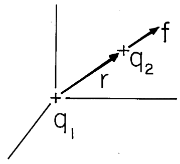

Fig. 1.3.9 Coulomb force induced on charge q_2 due to field from q_1.

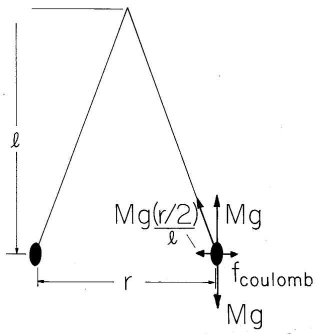

Fig. 1.3.10 Like-charged particles on ends of thread are pushed apart by the Coulomb force.

charge, and is inversely proportional to the square of the distance between them, is now demonstrated.

Demonstration 1.3.1. Coulomb's Force Law

The charge resulting on the surface of adhesive tape as it is pulled from a dispenser is a common nuisance. As the tape is brought toward a piece of paper, the force of attraction that makes the paper jump is an aggravating reminder that there are charges on the tape. Just how much charge there is on the tape can be approximately determined by means of the simple experiment shown in Fig. 1.3.10.

Two pieces of freshly pulled tape about 7 cm long are folded up into balls and stuck on the ends of a thread having a total length of about 20 cm. The middle of the thread is then tied up so that the charged balls of tape are suspended free to swing. (By electrostatic standards, our fingers are conductors, so the tape should be manipulated chopstick fashion by means of plastic rods or the like.) It is then easy to measure approximately l and r, as defined in the figure. The force of repulsion that separates the "balls" of tape is presumably predicted by (15). In Fig. 1.3.10, the vertical component of the tension in the thread must balance the gravitational force Mg (where g is the gravitational acceleration and M is the mass). It follows that the horizontal component of the thread tension balances the Coulomb force of repulsion.

$$\frac{q^2}{4\pi\epsilon_o r^2} = Mg\frac{(r/2)}{l} \Rightarrow q = \sqrt{\frac{Mgr^3 2\pi\epsilon_o}{l}} \tag{16}$$

As an example, tape balls having an area of $A = 14$ cm^2, (7 cm length of 2 cm wide tape) weighing 0.1 mg and dangling at a length $l = 20$ cm result in a distance of separation $r = 3$ cm. It follows from (16) (with all quantities expressed in SI units) that $q = 2.7 \times 10^{-9}$ coulomb. Thus, the average surface charge density is $q/A = 1.9 \times 10^{-6}$ coulomb/meter or 1.2×10^{13} electronic charges per square meter. If

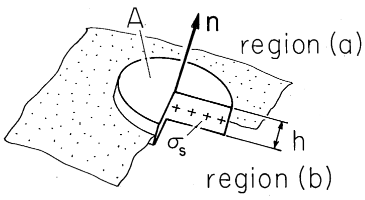

Fig. 1.3.11 Pillbox-shaped incremental volume used to deduce the jump condition implied by Gauss' integral law.

these charges were in a square array with spacing s between charges, then $\sigma_s = e/s^2$, and it follows that the approximate distance between the individual charge in the tape surface is $0.3\mu m$. This length is at the limit of an optical microscope and may seem small. However, it is about 1000 times larger than a typical atomic dimension.[2]

Gauss' Continuity Condition. Each of the integral laws summarized in this chapter implies a relationship between field variables evaluated on either side of a surface. These conditions are necessary for dealing with surface singularities in the field sources. Example 1.3.2 illustrates the jump in the normal component of **E** that accompanies a surface charge.

A surface that supports surface charge is pictured in Fig. 1.3.11, as having a unit normal vector directed from region (b) to region (a). The volume to which Gauss' integral law is applied has the pillbox shape shown, with endfaces of area A on opposite sides of the surface. These are assumed to be small enough so that over the area of interest the surface can be treated as plane. The height h of the pillbox is very small so that the cylindrical sideface of the pillbox has an area much smaller than A.

Now, let h approach zero in such a way that the two sides of the pillbox remain on opposite sides of the surface. The volume integral of the charge density, on the right in (1), gives $A\sigma_s$. This follows from the definition of the surface charge density, (11). The electric field is assumed to be finite throughout the region of the surface. Hence, as the area of the sideface shrinks to zero, so also does the contribution of the sideface to the surface integral. Thus, the displacement flux through the closed surface consists only of the contributions from the top and bottom surfaces. Applied to the pillbox, Gauss' integral law requires that

$$\boxed{\mathbf{n} \cdot (\epsilon_o \mathbf{E}^a - \epsilon_o \mathbf{E}^b) = \sigma_s} \tag{17}$$

where the area A has been canceled from both sides of the equation.

The contribution from the endface on side (b) comes with a minus sign because on that surface, $\mathbf{n}$ is opposite in direction to the surface element $d\mathbf{a}$.

Note that the field found in Example 1.3.2 satisfies this continuity condition at $z = s/2$ and $z = -s/2$.

[2] An alternative way to charge a particle, perhaps of low density plastic, is to place it in the corona discharge around the tip of a pin placed at high voltage. The charging mechanism at work in this case is discussed in Chapter 7 (Example 7.7.2).

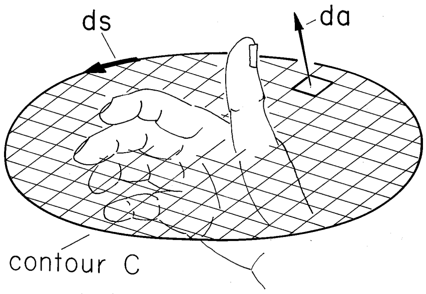

Fig. 1.4.1 Surface S is enclosed by contour C having positive direction determined by the right-hand rule. With the fingers in the direction of $d\mathbf{s}$, the thumb passes through the surface in the direction of positive $d\mathbf{a}$.

1.4 AMPÈRE'S INTEGRAL LAW

The law relating the *magnetic field intensity* $\mathbf{H}$ to its source, the current density $\mathbf{J}$, is

$$\boxed{\oint_C \mathbf{H}\cdot d\mathbf{s} = \int_S \mathbf{J}\cdot d\mathbf{a} + \frac{d}{dt}\int_S \epsilon_o \mathbf{E}\cdot d\mathbf{a}} \tag{1}$$

Note that by contrast with the integral statement of Gauss' law, (1.3.1), the surface integral symbols on the right do not have circles. This means that the integrations are over open surfaces, having edges denoted by the contour C. Such a surface S enclosed by a contour C is shown in Fig. 1.4.1. In words, Ampère's integral law as given by (1) requires that the line integral (circulation) of the magnetic field intensity $\mathbf{H}$ around a closed contour is equal to the net current passing through the surface spanning the contour plus the time rate of change of the net displacement flux density $\epsilon_o\mathbf{E}$ through the surface (the *displacement current*).

The direction of positive $d\mathbf{a}$ is determined by the right-hand rule, as also illustrated in Fig. 1.4.1. With the fingers of the right-hand in the direction of $d\mathbf{s}$, the thumb has the direction of $d\mathbf{a}$. Alternatively, with the right hand thumb in the direction of $d\mathbf{s}$, the fingers will be in the positive direction of $d\mathbf{a}$.

In Ampère's law, $\mathbf{H}$ appears without μ_o. This law therefore establishes the basic units of $\mathbf{H}$ as coulomb/(meter-second). In Sec. 1.1, the units of the flux density $\mu_o\mathbf{H}$ are defined by the Lorentz force, so the second empirical constant, the *permeability of free space*, is $\mu_o = 4\pi\times 10^{-7}$ henry/m (henry = volt sec/amp).

Example 1.4.1. Magnetic Field Due to Axisymmetric Current

A constant current in the z direction within the circular cylindrical region of radius R, shown in Fig. 1.4.2, extends from − infinity to + infinity along the z axis and is represented by the density

$$\mathbf{J} = \begin{cases} J_o\left(\frac{r}{R}\right); & r < R \\ 0; & r > R \end{cases} \tag{2}$$

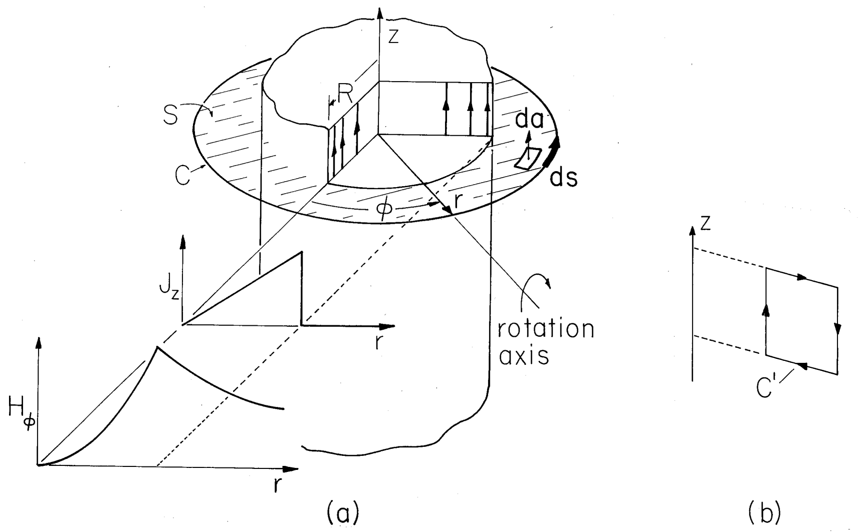

Fig. 1.4.2 Axially symmetric current distribution and associated radial distribution of azimuthal magnetic field intensity. Contour C is used to determine azimuthal $\mathbf{H}$, while C' is used to show that the z-directed field must be uniform.

where J_o and R are given constants. The associated magnetic field intensity has only an azimuthal component.

$$\mathbf{H} = H_\phi \mathbf{i}_\phi \tag{3}$$

To see that there can be no r component of this field, observe that rotation of the source around the radial axis, as shown in Fig. 1.4.2, reverses the source (the current is then in the $-z$ direction) and hence must reverse the field. But an r component of the field does not reverse under such a rotation and hence must be zero. The H_ϕ and H_z components are not ruled out by this argument. However, if they exist, they must not depend upon the ϕ and z coordinates, because rotation of the source around the z axis and translation of the source along the z axis does not change the source and hence does not change the field.

The current is independent of time and so we assume that the fields are as well. Hence, the last term in (1), the displacement current, is zero. The law is then used with S, a surface having its enclosing contour C at the arbitrary radius r, as shown in Fig. 1.4.2. Then the area and line elements are

$$d\mathbf{a} = r d\phi dr \mathbf{i}_z; \qquad d\mathbf{s} = \mathbf{i}_\phi r d\phi \tag{4}$$

and the right-hand side of (1) becomes

$$\int_S \mathbf{J} \cdot d\mathbf{a} = \begin{cases} \int_0^{2\pi} \int_0^r J_o \frac{r}{R} r d\phi dr = \frac{J_o r^3 2\pi}{3R}; & r < R \\ \int_0^{2\pi} \int_0^R J_o \frac{r}{R} r d\phi dr = \frac{J_o R^2 2\pi}{3}; & R < r \end{cases} \tag{5}$$

Integration on the left-hand side amounts to a multiplication of the ϕ independent H_ϕ by the length of C.

$$\oint_C \mathbf{H} \cdot d\mathbf{s} = \int_0^{2\pi} H_\phi r d\phi = H_\phi 2\pi r \tag{6}$$

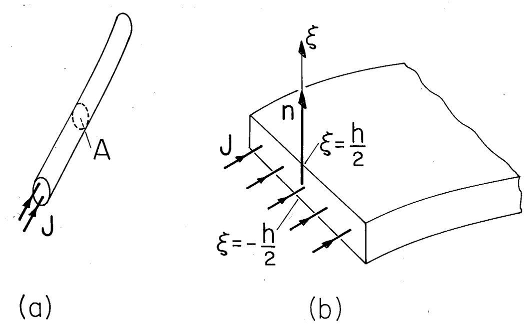

Fig. 1.4.3 (a) Line current enclosed by volume having cross-sectional area A. (b) Surface current density enclosed by contour having thickness h.

These last two expressions are used to evaluate (1) and obtain

$$2\pi r H_\phi = \frac{J_o r^3 2\pi}{3R} \Rightarrow H_\phi = \frac{J_o r^2}{3R}; \qquad r < R$$

$$2\pi r H_\phi = \frac{J_o R^2 2\pi}{3} \Rightarrow H_\phi = \frac{J_o R^2}{3r}; \qquad r < R \tag{7}$$

Thus, the azimuthal magnetic field intensity has the radial distribution shown in Fig. 1.4.2.

The z component of $\mathbf{H}$ is, at most, uniform. This can be seen by applying the integral law to the contour C', also shown in Fig. 1.4.2. Integration on the top and bottom legs gives zero because $H_r = 0$. Thus, to make the contributions due to H_z on the vertical legs cancel, it is necessary that H_z be independent of radius. Such a uniform field must be caused by sources at infinity and is therefore set equal to zero if such sources are not postulated in the statement of the problem.

Singular Current Distributions. The first of two singular forms of the current density shown in Fig. 1.4.3a is the *line current*. Formally, it is the limit of an infinite current density distributed over an infinitesimal area.

$$i = \lim_{\substack{|\mathbf{J}|\to\infty \\ A\to 0}} \int_A \mathbf{J}\cdot d\mathbf{a} \tag{8}$$

With i a constant over the length of the line, a thin wire carrying a current i conjures up the correct notion of the line current. However, in general, the current i may depend on the position along the line if it varies with time as in an antenna.

The second singularity, the *surface current density*, is the limit of a very large current density $\mathbf{J}$ distributed over a very thin layer adjacent to a surface. In Fig. 1.4.3b, the current is in a direction parallel to the surface. If the layer extends between $\xi = -h/2$ and $\xi = +h/2$, the *surface current density* $\mathbf{K}$ is defined as

$$\mathbf{K} = \lim_{\substack{|\mathbf{J}|\to\infty \\ h\to 0}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{J} d\xi \tag{9}$$

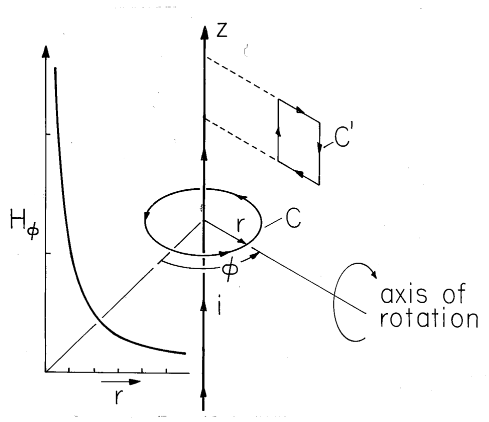

Fig. 1.4.4 Uniform line current with contours for determining **H**. Axis of rotation is used to deduce that radial component of field must be zero.

By definition, **K** is a vector tangential to the surface that has units of ampere/meter.

Illustration. *H* field Produced by a Uniform Line Current

A uniform line current of magnitude i extends from $-$ infinity to $+$ infinity along the z axis, as shown in Fig. 1.4.4. The symmetry arguments of Example 1.4.1 show that the only component of **H** is azimuthal. Application of Ampère's integral law, (1), to the contour of Fig. 1.4.4 having arbitrary radius r gives a line integral that is simply the product of H_ϕ and the circumference $2\pi r$ and a surface integral that is simply i, regardless of the radius.

$$2\pi r H_\phi = i \Rightarrow H_\phi = \frac{i}{2\pi r} \tag{10}$$

This expression makes it especially clear that the units of **H** are ampere/meter.

Demonstration 1.4.1. Magnetic Field of a Line Current

At 60 Hz, the displacement current contribution to the magnetic field of the experiment shown in Fig. 1.4.5 is negligible. So long as the field probe is within a distance r from the wire that is small compared to the distance to the ends of the wire or to the return wires below, the magnetic field intensity is predicted quantitatively by (10). The curve shown is typical of demonstration measurements illustrating the radial dependence. Because the Hall-effect probe fundamentally exploits the Lorentz force law, it measures the flux density $\mu_o\mathbf{H}$. A common unit for flux density is the Gauss. For conversion of units, 10,000 gauss = 1 tesla, where the tesla is the SI unit.

Illustration. Uniform Axial Surface Current

At the radius R from the z axis, there is a uniform z directed surface current density K_o that extends from - infinity to $+$ infinity in the z direction. The symmetry arguments of Example 1.4.1 show that the resulting magnetic field intensity

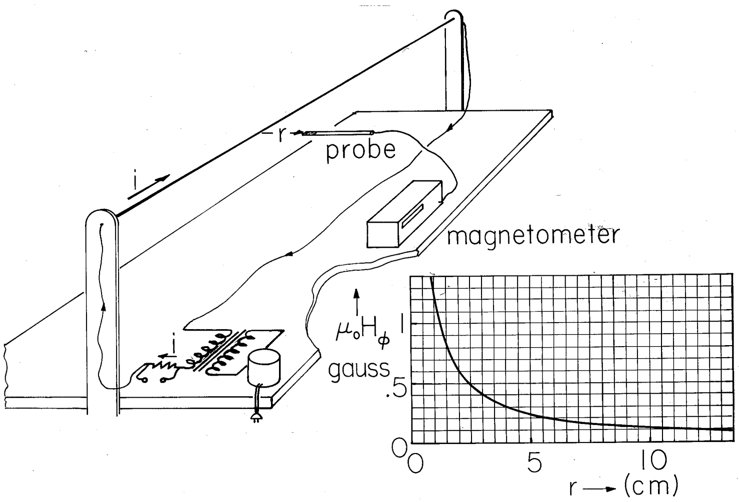

Fig. 1.4.5 Demonstration of peak magnetic flux density induced by line current of 6 ampere (peak).

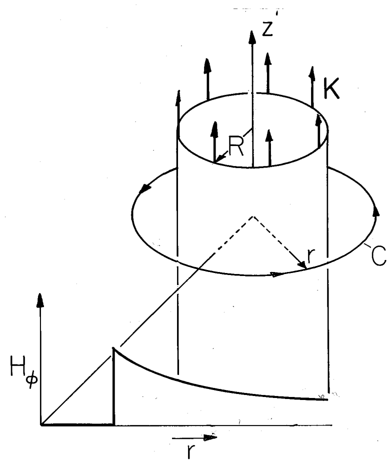

Fig. 1.4.6 Uniform current density K_o is z directed in circular cylindrical shell at $r = R$. Radially discontinuous azimuthal field shown is determined using the contour at arbitrary radius r.

is azimuthal. To determine that field, Ampère's integral law is applied to a contour having the arbitrary radius r, shown in Fig. 1.4.6. As in the previous illustration, the line integral is the product of the circumference and H_ϕ. The surface integral gives nothing if $r < R$, but gives $2\pi R$ times the surface current density if $r > R$. Thus,

$$2\pi r H_\phi = \begin{cases} 0; & r < R \\ 2\pi R K_o; & r > R \end{cases} \Rightarrow H_\phi = \begin{cases} 0; & r < R \\ K_o \frac{R}{r}; & r > R \end{cases} \tag{11}$$

Thus, the distribution of $\mathbf{H}_\phi$ is the discontinuous function shown in Fig. 1.4.6. The field tangential to the surface current undergoes a jump that is equal in magnitude

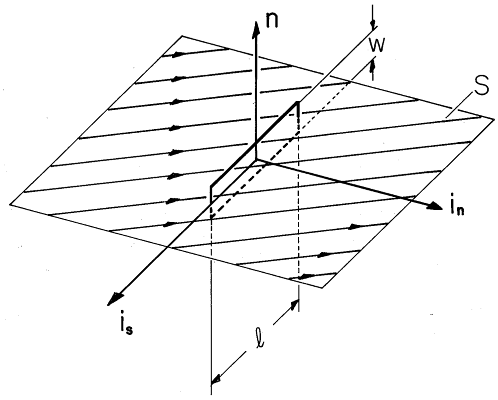

Fig. 1.4.7 Ampère's integral law is applied to surface S' enclosed by a rectangular contour that intersects a surface S carrying the current density $\mathbf{K}$. In terms of the unit normal to S, $\mathbf{n}$, the resulting continuity condition is given by (16).

to the surface current density.

Ampère's Continuity Condition. A surface current density in a surface S causes a discontinuity of the magnetic field intensity. This is illustrated in Fig. 1.4.6. To obtain a general relation between fields evaluated to either side of S, a rectangular surface of integration is mounted so that it intersects S as shown in Fig. 1.4.7. The normal to S is in the plane of the surface of integration. The length l of the rectangle is assumed small enough so that the surface of integration can be considered plane over this length. The width w of the rectangle is assumed to be much smaller than l . It is further convenient to introduce, in addition to the normal $\mathbf{n}$ to S, the mutually orthogonal unit vectors $\mathbf{i_s}$ and $\mathbf{i_n}$ as shown.

Now apply the integral form of Ampère's law, (1), to the rectangular surface of area lw. For the right-hand side we obtain

$$\int_{S'} \mathbf{J} \cdot d\mathbf{a} + \int_{S'} \frac{\partial}{\partial t} \epsilon_o \mathbf{E} \cdot d\mathbf{a} \simeq \mathbf{K} \cdot \mathbf{i}_n l \tag{12}$$

Only $\mathbf{J}$ gives a contribution, and then only if there is an infinite current density over the zero thickness of S, as required by the definition of the surface current density, (9). The time rate of change of a finite displacement flux density integrated over zero area gives zero, and hence there is no contribution from the second term.

The left-hand side of Ampère's law, (1), is a contour integral following the rectangle. Because w has been assumed to be very small compared with l, and $\mathbf{H}$ is assumed finite, no contribution is made by the two short sides of the rectangle. Hence,

$$l\, \mathbf{i}_s \cdot (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \cdot \mathbf{i}_n l \tag{13}$$

From Fig. 1.4.7, note that

$$\mathbf{i}_s = \mathbf{i}_n \times \mathbf{n} \tag{14}$$

The cross and dot can be interchanged in this scalar triple product without affecting the result (Appendix 1), so introduction of (14) into (13) gives

$$\mathbf{i}_n \cdot \mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{i}_n \cdot \mathbf{K} \tag{15}$$

Finally, note that the vector $\mathbf{i}_n$ is arbitrary so long as it lies in the surface S. Since it multiplies vectors tangential to the surface, it can be omitted.

$$\boxed{\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}} \tag{16}$$

There is a jump in the tangential magnetic field intensity as one passes through a surface current. Note that (16) gives a prediction consistent with what was found for the illustration in Fig. 1.4.6.

1.5 CHARGE CONSERVATION IN INTEGRAL FORM

Embedded in the laws of Gauss and Ampère is a relationship that must exist between the charge and current densities. To see this, first apply Ampère's law to a closed surface, such as sketched in Fig. 1.5.1. If the contour **C** is regarded as the"drawstring" and S as the "bag," then this limit is one in which the "string" is drawn tight so that the contour shrinks to zero. Thus, the open surface integrals of (1.4.1) become closed, while the contour integral vanishes.

$$\oint_S \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \oint_S \epsilon_o \mathbf{E} \cdot d\mathbf{a} = 0 \tag{1}$$

But now, in view of Gauss' law, the surface integral of the electric displacement can be replaced by the total charge enclosed. That is, (1.3.1) is used to write (1) as

$$\boxed{\oint_S \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_V \rho dv = 0} \tag{2}$$

This is the law of conservation of charge. If there is a net current out of the volume shown in Fig. 1.5.2, (2) requires that the net charge enclosed be decreasing with time.

Charge conservation, as expressed by (2), was a compelling reason for Maxwell to add the electric displacement term to Ampère's law. Without the displacement current density, Ampère's law would be inconsistent with charge conservation. That is, if the second term in (1) would be absent, then so would the second term in (2). If the displacement current term is dropped in Ampère's law, then net current cannot enter, or leave, a volume.

The conservation of charge is consistent with the intuitive picture of the relationship between charge and current developed in Example 1.2.1.

Example 1.5.1. Continuity of Convection Current

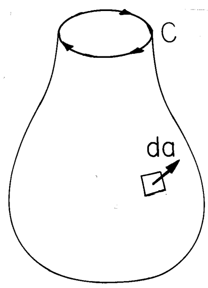

Fig. 1.5.1 Contour C enclosing an open surface can be thought of as the drawstring of a bag that can be closed to create a closed surface.

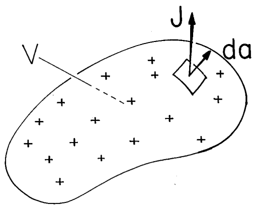

Fig. 1.5.2 Current density leaves a volume V and hence the net charge must decrease.

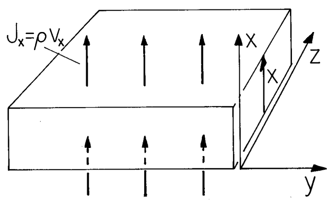

Fig. 1.5.3 In steady state, charge conservation requires that the current density entering through the $x = 0$ plane be the same as that leaving through the plane at $x = x$.

The steady state current of electrons accelerated through vacuum by a uniform electric field is described in Example 1.2.1 by assuming that in any plane $x =$ constant the current density is the same. That this must be true is now seen formally by applying the charge conservation integral theorem to the volume shown in Fig. 1.5.3. Here the lower surface is in the injection plane $x = 0$, where the current density is known to be J_o. The upper surface is at the arbitrary level denoted by x. Because the steady state prevails, the time derivative in (2) is zero. The remaining surface integral has contributions only from the top and bottom surfaces. Evaluation of these, with the recognition that the area element on the top surface is $(\mathbf{i_x} dydz)$ while it is $(-\mathbf{i_x} dydz)$ on the bottom surface, makes it clear that

$$AJ_x - AJ_o = 0 \Rightarrow \rho v_x = J_o \tag{3}$$

This same relation was used in Example 1.2.1, (1.2.4), as the basis for converting from a particle point of view to the one used here, where (x, y, z) are independent of t.

Example 1.5.2. Current Density and Time-Varying Charge

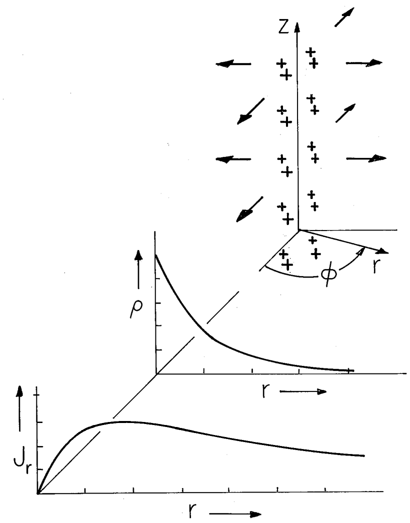

Fig. 1.5.4 With the given axially symmetric charge distribution positive and decreasing with time ($\partial\rho/\partial t < 0$), the radial current density is positive, as shown.

With the charge density a given function of time with an axially symmetric spatial distribution, (2) can be used to deduce the current density. In this example, the charge density is

$$\rho = \rho_o(t)e^{-r/a} \tag{4}$$

and can be pictured as shown in Fig. 1.5.4. The function of time ρ_o is given, as is the dimension a.

As the first step in finding $\mathbf{J}$, we evaluate the volume integral in (2) for a circular cylinder of radius r having z as its axis and length l in the z direction.

$$\begin{aligned}\int_V \rho dv &= \int_0^l \int_0^{2\pi} \int_0^r \rho_o e^{-\frac{r}{a}} dr(rd\phi)dz \\ &= 2\pi l a^2 \big[1 - e^{-\frac{r}{a}}\big(1 + \frac{r}{a}\big)\big]\rho_o\end{aligned} \tag{5}$$

The axial symmetry demands that $\mathbf{J}$ is in the radial direction and independent of ϕ and z. Thus, the evaluation of the surface integral in (2) amounts to a multiplication of J_r by the area $2\pi rl$, and that equation becomes

$$2\pi r l J_r + 2\pi l a^2\big[1 - e^{-\frac{r}{a}}\big(1 + \frac{r}{a}\big)\big]\frac{d\rho_o}{dt} = 0 \tag{6}$$

Finally, this expression can be solved for J_r.

$$J_r = \frac{a^2}{r}\big[e^{-\frac{r}{a}}\big(1 + \frac{r}{a}\big) - 1\big]\frac{d\rho_o}{dt} \tag{7}$$

Under the assumption that the charge density is positive and decreasing, so that $d\rho_o/dt < 0$, the radial distribution of J_r is shown at an instant in time in

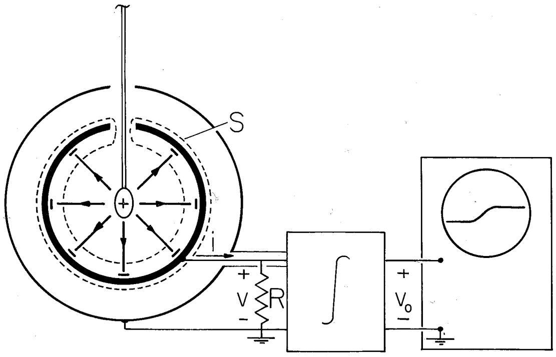

Fig. 1.5.5 When a charge q is introduced into an essentially grounded metal sphere, a charge $-q$ is induced on its inner surface. The integral form of charge conservation, applied to the surface S, shows that $i = dq/dt$. The net excursion of the integrated signal is then a direct measurement of q.

Fig. 1.5.4. In this case, the radial current density is positive at any radius r because the net charge within that radius, given by (5), is decreasing with time.

The integral form of charge conservation provides the link between the current carried by a wire and the charge. Thus, if we can measure a current, this law provides the basis for measuring the net charge. The following demonstration illustrates its use.

Demonstration 1.5.1. Measurement of Charge

In Demonstration 1.3.1, the net charge is deduced from mechanical measurements and Coulomb's force law. Here that same charge is deduced electrically. The "ball" carrying the charge is stuck to the end of a thin plastic rod, as in Fig. 1.5.5. The objective is to measure this charge, q, without removing it from the ball.

We know from the discussion of Gauss' law in Sec. 1.3 that this charge is the source of an electric field. In general, this field terminates on charges of opposite sign. Thus, the net charge that terminates the field originating from q is equal in magnitude and opposite in sign to q. Measurement of this "image" charge is tantamount to measuring q.

How can we design a metal electrode so that we are guaranteed that all of the lines of $\mathbf{E}$ originating from q will be terminated on its surface? It would seem that the electrode should essentially surround q. Thus, in the experiment shown in Fig. 1.5.5, the charge is transported to the interior of a metal sphere through a hole in its top. This sphere is grounded through a resistance R and also surrounded by a grounded shield. This resistance is made low enough so that there is essentially no electric field in the region between the spherical electrode, and the surrounding shield. As a result, there is negligible charge on the outside of the electrode and the net charge on the spherical electrode is just that inside, namely $-q$.

Now consider the application of (2) to the surface S shown in Fig. 1.5.5. The surface completely encloses the spherical electrode while excluding the charge q at its center. On the outside, it cuts through the wire connecting the electrode to the resistance R. Thus, the volume integral in (2) gives the net charge $-q$, while contributions to the surface integral only come from where S cuts through the wire. By definition, the integral of $\mathbf{J}\cdot d\mathbf{a}$ over the cross-section of the wire gives the current i (amps). Thus, (2) becomes simply

$$i + \frac{d(-q)}{dt} = 0 \Rightarrow i = \frac{dq}{dt} \tag{8}$$

This current is the result of having pushed the charge through the hole to a position where all the field lines terminated on the spherical electrode.[3]

Although small, the current through the resistor results in a voltage.

$$v \simeq iR = R\frac{dq}{dt} \tag{9}$$

The integrating circuit is introduced into the experiment in Fig. 1.5.5 so that the oscilloscope directly displays the charge. With this circuit goes a gain A such that

$$v_o = A\int v dt = ARq \tag{10}$$

Then, the voltage v_o to which the trace on the scope rises as the charge is inserted through the hole reflects the charge q. This measurement of q corroborates that of Demonstration 1.3.1.

In retrospect, because S and V are arbitrary in the integral laws, the experiment need not be carried out using an electrode and shield that are spherical. These could just as well have the shape of boxes.

Charge Conservation Continuity Condition. The continuity condition associated with charge conservation can be derived by applying the integral law to the same pillbox-shaped volume used to derive Gauss' continuity condition, (1.3.17). It can also be found by simply recognizing the similarity between the integral laws of Gauss and charge conservation. To make this similarity clear, rewrite (2) putting the time derivative under the integral. In doing so, d/dt must again be replaced by $\partial/\partial t$, because the time derivative now operates on ρ, a function of t and $\mathbf{r}$.

$$\oint_S \mathbf{J}\cdot d\mathbf{a} + \int_V \frac{\partial \rho}{\partial t} dV = 0 \tag{11}$$

Comparison of (11) with Gauss' integral law, (1.3.1), shows the similarity. The role of $\epsilon_o\mathbf{E}$ in Gauss' law is played by $\mathbf{J}$, while that of ρ is taken by $-\partial\rho/\partial t$. Hence, by analogy with the continuity condition for Gauss' law, (1.3.17), the continuity condition for charge conservation is

[3] Note that if we were to introduce the charged ball without having the spherical electrode essentially grounded through the resistance R, charge conservation (again applied to the surface S) would require that the electrode retain charge neutrality. This would mean that there would be a charge q on the outside of the electrode and hence a field between the electrode and the surrounding shield. With the charge at the center and the shield concentric with the electrode, this outside field would be the same as in the absence of the electrode, namely the field of a point charge, (1.3.12).

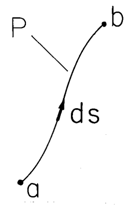

Fig. 1.6.1 Integration line for definition of electromotive force.

$$\boxed{\mathbf{n}\cdot(\mathbf{J}^a-\mathbf{J}^b)+\frac{\partial\sigma_s}{\partial t}=0} \tag{12}$$

Implicit in this condition is the assumption that $\mathbf{J}$ is finite. Thus, the condition does not include the possibility of a surface current.

1.6 FARADAY'S INTEGRAL LAW

The laws of Gauss and Ampère relate fields to sources. The statement of charge conservation implied by these two laws relates these sources. Thus, the previous three sections either relate fields to their sources or interrelate the sources. In this and the next section, integral laws are introduced that do not involve the charge and current densities.

Faraday's integral law states that the circulation of $\mathbf{E}$ around a contour C is determined by the time rate of change of the magnetic flux linking the surface enclosed by that contour (the magnetic induction).

$$\boxed{\oint_C \mathbf{E}\cdot d\mathbf{s}=-\frac{d}{dt}\int_S \mu_o\mathbf{H}\cdot d\mathbf{a}} \tag{1}$$

As in Ampère's integral law and Fig. 1.4.1, the right-hand rule relates $d\mathbf{s}$ and $d\mathbf{a}$.

The *electromotive force*, or EMF, between points (a) and (b) along the path P shown in Fig. 1.6.1 is defined as

$$\mathcal{E}_{ab}=\int_{(a)}^{(b)}\mathbf{E}\cdot\mathbf{ds} \tag{2}$$

We will accept this definition for now and look forward to a careful development of the circumstances under which the EMF is measured as a voltage in Chaps. 4 and 10.

Electric Field Intensity with No Circulation. First, suppose that the time rate of change of the magnetic flux is negligible, so that the electric field is essentially

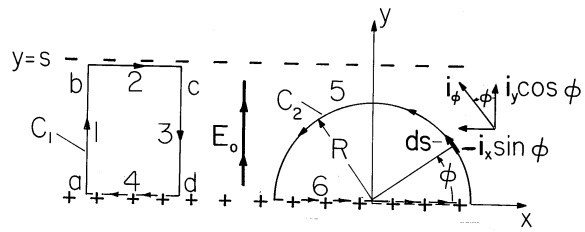

Fig. 1.6.2 Uniform electric field intensity E_o, between plane parallel uniform distributions of surface charge density, has no circulation about contours C_1 and C_2.

free of circulation. This means that no matter what closed contour C is chosen, the line integral of $\mathbf{E}$ must vanish.

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = 0 \qquad (3)$$

We will find that this condition prevails in electroquasistatic systems and that all of the fields in Sec. 1.3 satisfy this requirement.

Illustration. A Field Having No Circulation

A static field between plane parallel sheets of uniform charge density has no circulation. Such a field, $\mathbf{E} = E_o\mathbf{i_x}$, exists in the region $0 < y < s$ between the sheets of surface charge density shown in Fig. 1.6.2. The most convenient contour for testing this claim is denoted C_1 in Fig. 1.6.2.

Along path 1, $\mathbf{E} \cdot d\mathbf{s} = E_o dy$, and integration from $y = 0$ to $y = s$ gives sE_o for the EMF of point (a) relative to point (b). Note that the EMF between the plane parallel surfaces in Fig. 1.6.2 is the same regardless of where the points (a) and (b) are located in the respective surfaces.

On segments 2 and 4, $\mathbf{E}$ is orthogonal to $d\mathbf{s}$, so there is no contribution to the line integral on these two sections. Because $d\mathbf{s}$ has a direction opposite to $\mathbf{E}$ on segment 3, the line integral is the integral from $y = 0$ to $y = s$ of $\mathbf{E} \cdot d\mathbf{s} = -E_o dy$. The result of this integration is $-sE_o$, so the contributions from segments 1 and 3 cancel, and the circulation around the closed contour is indeed zero.[4]

In this planar geometry, a field that has only a y component cannot be a function of x without incurring a circulation. This is evident from carrying out this integration for such a field on the rectangular contour C_1. Contributions to paths 1 and 3 cancel only if $\mathbf{E}$ is independent of x.

Example 1.6.1. Contour Integration

To gain some appreciation for what it means to require of $\mathbf{E}$ that it have no circulation, no matter what contour is chosen, consider the somewhat more complicated contour C_2 in the uniform field region of Fig. 1.6.2. Here, C_2 is composed of the

[4] In setting up the line integral on a contour such as 3, which has a direction opposite to that in which the coordinate increases, it is tempting to double-account for the direction of $d\mathbf{s}$ not only be recognizing that $d\mathbf{s} = -\mathbf{i_y} dy$, but by integrating from $y = s$ to $y = 0$ as well.

semicircle (5) and the straight segment (6). On the latter, $\mathbf{E}$ is perpendicular to $d\mathbf{s}$ and so there is no contribution there to the circulation.

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = \int_5 \mathbf{E} \cdot d\mathbf{s} + \int_6 \mathbf{E} \cdot d\mathbf{s} = \oint_5 \mathbf{E} \cdot d\mathbf{s} \tag{4}$$

On segment 5, the vector differential $d\mathbf{s}$ is first written in terms of the unit vector $\mathbf{i}_\phi$, and that vector is in turn written (with the help of the vector decomposition shown in the figure) in terms of the Cartesian unit vectors.

$$d\mathbf{s} = \mathbf{i}_\phi R d\phi; \qquad \mathbf{i}_\phi = \mathbf{i}_y \cos\phi - \mathbf{i}_x \sin\phi \tag{5}$$

It follows that on the segment 5 of contour C_2

$$\mathbf{E} \cdot d\mathbf{s} = E_o \cos\phi R d\phi \tag{6}$$

and integration gives

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = \int_0^\pi E_o \cos\phi R d\phi = [E_o R \sin\phi]_0^\pi = 0 \tag{7}$$

So for contour C_2, the circulation of $\mathbf{E}$ is also zero.

When the electromotive force between two points is path independent, we call it the voltage between the two points. For a field having no circulation, the EMF must be independent of path. This we will recognize formally in Chap. 4.

Electric Field Intensity with Circulation. The second limiting situation, typical of the magnetoquasistatic systems to be considered, is primarily concerned with the circulation of $\mathbf{E}$, and hence with the part of the electric field generated by the time-varying magnetic flux density. The remarkable fact is that Faraday's law holds for any contour, whether in free space or in a material. Often, however, the contour of interest coincides with a conducting wire, which comprises a coil that links a magnetic flux density.

Illustration. Terminal EMF of a Coil

A coil with one turn is shown in Fig. 1.6.3. Contour (1) is inside the wire, while (2) joins the terminals along a defined path. With these contours constituting C, Faraday's integral law as given by (1) determines the terminal electromotive force. If the electrical resistance of the wire can be regarded as zero, in the sense that the electric field intensity inside the wire is negligible, the contour integral reduces to an integration from (b) to (a).[5] In view of the definition of the EMF, (2), this integration gives the negative of the EMF. Thus, Faraday's law gives the terminal EMF as

$$\mathcal{E}_{ab} = \frac{d}{dt}\lambda_f; \qquad \lambda_f \equiv \int_S \mu_o \mathbf{H} \cdot d\mathbf{a} \tag{8}$$

[5] With the objectives here limited to attaching an intuitive meaning to Faraday's law, we will give careful attention to the conditions required for this terminal relation to hold in Chaps. 8, 9, and 10.

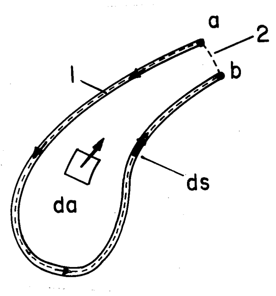

Fig. 1.6.3 Line segment (1) through a perfectly conducting wire and (2) joining the terminals (a) and (b) form closed contour.

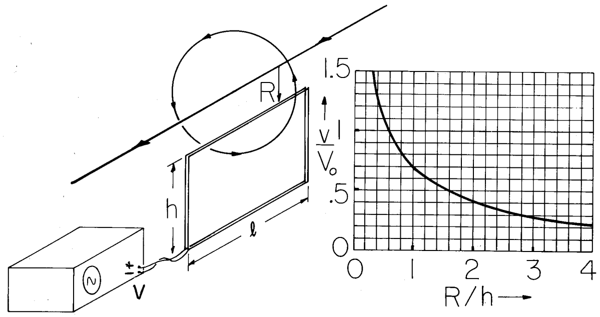

Fig. 1.6.4 Demonstration of voltmeter reading induced at terminals of a coil in accordance with Faraday's law. To plot data on graph, normalize voltage to V_o as defined with (11). Because I is the peak current, v is the peak voltage.

where λ_f, the total flux of magnetic field linking the coil, is defined as the flux linkage. Note that Faraday's law makes it possible to measure $\mu_o\mathbf{H}$ electrically (as now demonstrated).

Demonstration 1.6.1. Voltmeter Reading Induced by Magnetic Induction

The rectangular coil shown in Fig. 1.6.4 is used to measure the magnetic field intensity associated with current in a wire. Thus, the arrangement and field are the same as in Demonstration 1.4.1. The height and length of the coil are h and l as shown, and because the coil has N turns, it links the flux enclosed by one turn N times. With the upper conductors of the coil at a distance R from the wire, and the magnetic field intensity taken as that of a line current, given by (1.4.10), evaluation of (8) gives

$$\lambda_f = \mu_o N \int_z^{z+l} \int_R^{R+h} \frac{i}{2\pi r} dr dz = \left[\frac{\mu_o N l}{2\pi} ln\left(1 + \frac{h}{R}\right)\right] i \tag{9}$$

In the experiment, the current takes the form

$$i = I \sin \omega t \tag{10}$$

where $\omega = 2\pi(60)$. The EMF between the terminals then follows from (8) and (9) as

$$v = V_o ln\Big(1 + \frac{h}{R}\Big)\cos\omega t; \qquad V_o \equiv \frac{\mu_o N l \omega I}{2\pi} \tag{11}$$

A voltmeter reads the electromotive force between the two points to which it is connected, provided certain conditions are satisfied. We will discuss these in Chap. 8.

In a typical experiment using a 20-turn coil with dimensions of $h = 8$ cm, $l = 20$ cm, $I = 6$ amp peak, the peak voltage measured at the terminals with a spacing $R = 8$ cm is $v = 1.35$ mV. To put this data point on the normalized plot of Fig. 1.6.4, note that $R/h = 1$ and the measured $v/V_o = 0.7$.

Faraday's Continuity Condition. It follows from Faraday's integral law that the tangential electric field is continuous across a surface of discontinuity, provided that the magnetic field intensity is finite in the neighborhood of the surface of discontinuity. This can be shown by applying the integral law to the incremental surface shown in Fig. 1.4.7, much as was done in Sec. 1.4 for Ampère's law. With $\mathbf{J}$ set equal to zero, there is a formal analogy between Ampère's integral law, (1.4.1), and Faraday's integral law, (1). The former becomes the latter if $\mathbf{H} \rightarrow \mathbf{E}$, $\mathbf{J} \rightarrow \mathbf{0}$, and $\epsilon_o\mathbf{E} \rightarrow -\mu_o\mathbf{H}$. Thus, Ampère's continuity condition (1.4.16) becomes the continuity condition associated with Faraday's law.

$$\boxed{\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0} \tag{12}$$

At a surface having the unit normal $\mathbf{n}$, the tangential electric field intensity is continuous.

1.7 GAUSS' INTEGRAL LAW OF MAGNETIC FLUX

The net magnetic flux out of any region enclosed by a surface S must be zero.

$$\boxed{\oint_S \mu_o \mathbf{H} \cdot d\mathbf{a} = 0} \tag{1}$$

This property of flux density is almost implicit in Faraday's law. To see this, consider that law, (1.6.1), applied to a closed surface S. Such a surface is obtained from an open one by letting the contour shrink to zero, as in Fig. 1.5.1. Then Faraday's integral law reduces to

$$\frac{d}{dt}\oint_S \mu_o \mathbf{H} \cdot \mathbf{da} = \mathbf{0} \tag{2}$$

Gauss' law (1) adds to Faraday's law the empirical fact that in the beginning, there was no closed surface sustaining a net outward magnetic flux.

Illustration. Uniqueness of Flux Linking Coil

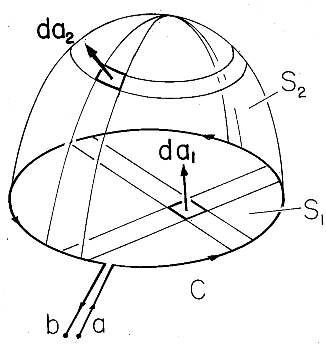

Fig. 1.7.1 Contour C follows loop of wire having terminals $a - b$. Because each has the same enclosing contour, the net magnetic flux through surfaces S_1 and S_2 must be the same.

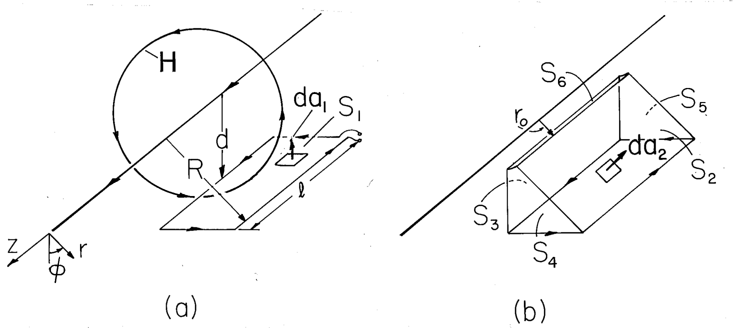

Fig. 1.7.2 (a) The field of a line current induces a flux in a horizontal rectangular coil. (b) The open surface has the coil as an enclosing contour. Rather than being in the plane of the contour, this surface is composed of the five segments shown.

An example is shown in Fig. 1.7.1. Here a wire with terminals $a - b$ follows the contour C. According to (1.6.8), the terminal EMF is found by integrating the normal magnetic flux density over a surface having C as its edge. But which surface? Figure 1.7.1 shows two of an infinite number of possibilities.

The terminal EMF can be unique only if the integrals over S_1 and S_2 result in the same answer. Taken together, S_1 and S_2 form a closed surface. The magnetic flux continuity integral law, (1), requires that the net flux out of this closed surface be zero. This is equivalent to the statement that the flux passing through S_1 in the direction of $d\mathbf{a}_1$ must be equal to that passing through S_2 in the direction of $d\mathbf{a}_2$. We will formalize this statement in Chap. 8.

Example 1.7.1. Magnetic Flux Linked by Coil and Flux Continuity

In the configuration of Fig. 1.7.2, a line current produces a magnetic field intensity that links a one-turn coil. The left conductor in this coil is directly below the wire at a distance d. The plane of the coil is horizontal. Nevertheless, it is convenient to specify the position of the right conductor in terms of a distance R from the line current. What is the net flux linked by the coil?

The most obvious surface to use is one in the same plane as the coil. However, in doing so, account must be taken of the way in which the unit normal to the surface varies in direction relative to the magnetic field intensity. Selection of another surface, to which the magnetic field intensity is either normal or tangential, simplifies the calculation. On surfaces S_2 and S_3, the normal direction is the direction of the magnetic field. Note also that because the field is tangential to the end surfaces, S_4 and S_5, these make no contribution. For the same reason, there is no contribution from S_6, which is at the radius r_o from the wire. Thus,

$$\lambda_f \equiv \int_S \mu_o \mathbf{H} \cdot d\mathbf{a} = \int_{S_2} \mu_o \mathbf{H} \cdot d\mathbf{a} + \int_{S_3} \mu_o \mathbf{H} \cdot d\mathbf{a} \tag{3}$$

On S_2 the unit normal is $\mathbf{i}_\phi$, while on S_3 it is $-\mathbf{i}_\phi$. Therefore, (3) becomes

$$\lambda_f = \int_0^l \int_{r_o}^R \mu_o H_\phi dr dz - \int_0^l \int_{r_o}^d \mu_o H_\phi dr dz \tag{4}$$

With the field intensity for a line current given by (1.4.10), it follows that

$$\lambda_f = \frac{\mu_o l i}{2\pi}\big(ln\frac{R}{r_o} - ln\frac{d}{r_o}\big) = \mu_o \frac{li}{2\pi} ln\big(\frac{R}{d}\big) \tag{5}$$

That r_o does not appear in the answer is no surprise, because if the surface S_1 had been used, r_o would not have been brought into the calculation.

Magnetic Flux Continuity Condition. With the charge density set equal to zero, the magnetic continuity integral law (1) takes the same form as Gauss' integral law (1.3.1). Thus, Gauss' continuity condition (1.3.17) becomes one representing the magnetic flux continuity law by making the substitution $\epsilon_o \mathbf{E} \rightarrow \mu_\mathbf{o} \mathbf{H}$.

$$\boxed{\mathbf{n} \cdot (\mu_o \mathbf{H}^a - \mu_o \mathbf{H}^b) = 0} \tag{6}$$

The magnetic flux density normal to a surface is continuous.

1.8 SUMMARY

Electromagnetic fields, whether they be inside a transistor, on the surfaces of an antenna or in the human nervous system, are defined in terms of the forces they produce. In every example involving electromagnetic fields, charges are moving somewhere in response to electromagnetic fields. Hence, our starting point in this introductory chapter is the Lorentz force on an elementary charge, (1.1.1). Represented by this law is the effect of the field on the charge and current (charge in motion).

The subsequent sections are concerned with the laws that predict how the field sources, the charge, and current densities introduced in Sec. 1.2, in turn give rise to the electric and magnetic fields. Our presentation is aimed at putting these laws to work. Hence, the empirical origins of these laws that would be evident from a historical presentation might not be fully appreciated. Elegant as they appear, Maxwell's equations are no more than a summary of experimental results. Each of our case studies is a potential test of the basic laws.

In the interest of being able to communicate our subject, each of the basic laws is given a name. In the interest of learning our subject, each of these laws should now be memorized. A summary is given in Table 1.8.1. By means of the examples and demonstrations, each of these laws should be associated with one or more physical consequences.

From the Lorentz force law and Maxwell's integral laws, the units of variables and constants are established. For the SI units used here, these are summarized in Table 1.8.2. Almost every practical result involves the free space permittivity ϵ_o and/or the free space permeability μ_o. Although these are summarized in Table 1.8.2, confidence also comes from having these natural constants memorized.

A common unit for measuring the magnetic flux density is the Gauss, so the conversion to the SI unit of Tesla is also given with the abbreviations.

A goal in this chapter has also been the use of examples to establish the mathematical significance of volume, surface, and contour integrations. At the same time, important singular source distributions have been defined and their associated fields derived. We will make extensive use of point, line, and surface sources and the associated fields.

In dealing with surface sources, a continuity condition should be associated with each of the integral laws. These are summarized in Table 1.8.3.

The continuity conditions should always be associated with the integral laws from which they originate. As terms are added to the integral laws to account for macroscopic media, there will be corresponding changes in the continuity conditions.

REFERENCES

[1] M. Faraday, **Experimental Researches in Electricity**, R. Taylor Publisher (1st-9th series), 1832-1835, 1 volume, various pagings; "From the Philosophical Transactions 1832-1835," London, England.

[2] J.C. Maxwell, **A Treatise on Electricity and Magnetism**, 3rd ed., 1891, reissued by Dover, N.Y. (1954).

TABLE 1.8.1

SUMMARY OF MAXWELL'S INTEGRAL LAWS IN FREE SPACE

NAME	INTEGRAL LAW	EQ. NUMBER
Gauss' Law	$\oint_S \epsilon_o \mathbf{E} \cdot d\mathbf{a} = \int_V \rho dv$	1.3.1
Ampere's Law	$\oint_C \mathbf{H} \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_S \epsilon_o \mathbf{E} \cdot d\mathbf{a}$	1.4.1
Faraday's Law	$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mu_o \mathbf{H} \cdot d\mathbf{a}$	1.6.1
Magnetic Flux Continuity	$\oint_S \mu_o \mathbf{H} \cdot d\mathbf{a} = 0$	1.7.1
Charge Conservation	$\oint_S \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_V \rho dv = 0$	1.5.2

TABLE 1.8.2

DEFINITIONS AND UNITS OF FIELD VARIABLES AND CONSTANTS

(basic unit of mass, kg, is replaced by V-C-s^2/m^2)

VARIABLE OR PARAMETER	NOMENCLATURE	BASIC UNITS	DERIVED UNITS
Electric Field Intensity	$\mathbf{E}$	V/m	V/m
Electric Displacement Flux Density	$\epsilon_o \mathbf{E}$	C/m^2	C/m^2
Charge Density	ρ	C/m^3	C/m^3
Surface Charge Density	σ_s	C/m^2	C/m^2
Magnetic Field Intensity	$\mathbf{H}$	C/(ms)	A/m
Magnetic Flux Density	$\mu_o \mathbf{H}$	Vs/m^2	T
Current Density	$\mathbf{J}$	$C/(m^2 s)$	A/m^2
Surface Current Density	$\mathbf{K}$	C/(ms)	A/m
Free Space Permittivity	$\epsilon_o = 8.854 \times 10^{-12}$	C/(Vm)	F/m
Free Space Permeability	$\mu_o = 4\pi \times 10^{-7}$	$Vs^2/(Cm)$	H/m

UNIT ABBREVIATIONS

Ampère	A	Kilogram	kg	Volt	V
Coulomb	C	Meter	m		
Farad	F	Second	s		
Henry	H	Tesla	T (10^4 Gauss)		

TABLE 1.8.3

SUMMARY OF CONTINUITY CONDITIONS IN FREE SPACE

NAME	CONTINUITY CONDITION	EQ. NUMBER
Gauss' Law	$\mathbf{n}\cdot(\epsilon_o\mathbf{E}^a-\epsilon_o\mathbf{E}^b)=\sigma_s$	1.3.17
Ampère's Law	$\mathbf{n}\times(\mathbf{H}^a-\mathbf{H}^b)=\mathbf{K}$	1.4.16
Faraday's Law	$\mathbf{n}\times(\mathbf{E}^a-\mathbf{E}^b)=0$	1.6.14
Magnetic Flux Continuity	$\mathbf{n}\cdot(\mu_o\mathbf{H}^a-\mu_o\mathbf{H}^b)=0$	1.7.6
Charge Conservation	$\mathbf{n}\cdot(\mathbf{J}^a-\mathbf{J}^b)+\frac{\partial\sigma_s}{\partial t}=0$	1.5.12

PROBLEMS

1.1 The Lorentz Law in Free Space*

1.1.1* Assuming in Example 1.1.1 that $v_i = 0$ and that $E_x < 0$, show that by the time the electron has reached the position $x = h$, its velocity is $\sqrt{-2eE_x h/m}$. In an electric field of only $E_x = 1\text{v/cm} = 10^{-2}$ v/m, show that by the time it reaches $h = 10^{-2}$ m, the electron has reached a velocity of 5.9×10^3 m/s.

1.1.2 An electron moves in vacuum under the same conditions as in Example 1.1.1 except that the electric field takes the form $\mathbf{E} = E_x\mathbf{i_x} + E_y\mathbf{i_y}$ where E_x and E_y are given constants. When $t = 0$, the electron is at $\xi_x = 0$ and $\xi_y = 0$ and the velocity $d\xi_x/dt = v_i$ and $d\xi_y/dt = 0$.

(a) Determine $\xi_x(t)$ and $\xi_y(t)$.
(b) For $E_x > 0$, when and where does the electron return to the plane $x = 0$?

1.1.3* An electron, having velocity $\mathbf{v} = v_i\mathbf{i_z}$, experiences the field $\mathbf{H} = H_o\mathbf{i_y}$ and $\mathbf{E} = E_o\mathbf{i_x}$, where H_o and E_o are constants. Show that the electron retains this velocity if $E_o = v_i\mu_o H_o$.

1.1.4 An electron has the initial position $x = 0,\ y = 0,\ z = z_o$. It has an initial velocity $\mathbf{v} = v_o\mathbf{i_x}$ and moves in the uniform and constant fields $\mathbf{E} = E_o\mathbf{i_y},\ \mathbf{H} = H_o\mathbf{i_y}$.

(a) Determine the position of the electron in the y direction, $\xi_y(t)$.
(b) Describe the trajectory of the electron.

1.2 Charge and Current Densities

1.2.1* The charge density is $\rho_o r/R$ coulomb/m^3 throughout the volume of a spherical region having radius R, with ρ_o a constant and r the distance from the center of the region (the radial coordinate in spherical coordinates). Show that the total charge associated with this charge density is $q = \pi\rho_o R^3$ coulomb.

1.2.2 In terms of given constants ρ_o and a, the net charge density is $\rho = (\rho_o/a^2)(x^2 + y^2 + z^2)$ coulomb/m^3. What is the total charge q (coulomb) in the cubical region $-a < x < a,\ -a < y < a,\ -a < z < a$?

* An asterisk on a problem number designates a "show that" problem. These problems are especially designed for self study.

1.2.3* With J_o and a given constants, the current density is $\mathbf{J} = (J_o/a^2)(y^2 + z^2)[\mathbf{i_x} + \mathbf{i_y} + \mathbf{i_z}]$. Show that the total current i passing through the surface $x = 0,\ -a < y < a,\ -a < z < a$ is $i = 8J_o a^2/3$ amp.

1.2.4 In cylindrical coordinates (r, ϕ, z) the current density is given in terms of constants J_o and a by $\mathbf{J} = J_o(r/a)^2\mathbf{i_z}$ (amp/m^2). What is the net current i (amp) through the surface $z = 0,\ r < a$?

1.2.5* In cylindrical coordinates, the electric field in the annular region $b < r < a$ is $\mathbf{E} = \mathbf{i_r}E_o(b/r)$, where E_o is a given negative constant. When $t = 0$, an electron having mass m and charge $q = -e$ has no velocity and is positioned at $r = \xi_r = b$.

(a) Show that, in vacuum, the radial motion of the electron is governed by the differential equation $mdv_r/dt = -eE_ob/\xi_r$, where $v_r = d\xi_r/dt$. Note that these expressions combine to provide one second-order differential equation governing ξ_r.

(b) By way of providing one integration of this equation, multiply the first of the first-order expressions by v_r and (with the help of the second first-order expression) show that the resulting equation can be written as $d[\frac{1}{2}mv_r^2 + eE_ob\ ln\xi_r]/dt = 0$. That is, the sum of the kinetic and potential energies (the quantity in brackets) remains constant.

(c) Use the result of (b) to find the electron velocity $v_r(r)$.

(d) Assume that this is one of many electrons that flow radially outward from the cathode at $r = b$ to $r = a$ and that the number of electrons passing radially outward at any location r is independent of time. The system is in the steady state so that the net current flowing outward through a surface of radius r and length l, $i = 2\pi rlJ_r$, is the same at one radius r as at another. Use this fact to determine the charge density $\rho(r)$.

1.3 Gauss' Integral Law

1.3.1* Consider how Gauss' integral law, (1), is evaluated for a surface that is not naturally symmetric. The charge distribution is the uniform line charge of Fig. 1.3.7 and hence $\mathbf{E}$ is given by (13). However, the surface integral on the left in (1) is to be evaluated using a surface that has unit length in the z direction and a square cross-section centered on the z axis. That is, the surface is composed of the planes $z = 0,\ z = 1,\ x = \pm a$, and $y = \pm a$. Thus, we know from evaluation of the right-hand side of (1) that evaluation of the surface integral on the left should give the line charge density λ_l.

(a) Show that the area elements $d\mathbf{a}$ on these respective surfaces are $\pm\mathbf{i_z}dxdy$, $\pm\mathbf{i_x}dydz$, and $\pm\mathbf{i_y}dxdz$.

(b) Starting with (13), show that in Cartesian coordinates, $\mathbf{E}$ is

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o}\left(\frac{x}{x^2+y^2}\mathbf{i_x} + \frac{y}{x^2+y^2}\mathbf{i_y}\right) \tag{a}$$

(Standard Cartesian and cylindrical coordinates are defined in Table I at the end of the text.)

(c) Show that integration of $\epsilon_o\mathbf{E}\cdot d\mathbf{a}$ over the part of the surface at $x = a$ leads to the integral

$$\int \epsilon_o\mathbf{E}\cdot d\mathbf{a} = \frac{\lambda_l}{2\pi}\int_0^1\int_{-a}^{a}\frac{a}{a^2+y^2}dydz \tag{b}$$

(d) Finally, show that integration over the entire closed surface indeed gives λ_l.

1.3.2 Using the spherical symmetry and a spherical surface, the electric field associated with the point charge q of Fig. 1.3.6 is found to be given by (12). Evaluation of the left-hand side of (1) over any other surface that encloses the point charge must also give q. Suppose that the closed surface S is composed of a hemisphere of radius a in the upper half-plane, a hemisphere of radius b in the lower half-plane, and a washer-shaped flat surface that joins the two. In spherical coordinates (defined in Table I), these three parts of the closed surface S are defined by $(r = a,\ 0 < \theta < \frac{1}{2}\pi,\ 0 \le \phi < 2\pi)$, $(r = b, \frac{1}{2}\pi < \theta < \pi,\ 0 \le \phi < 2\pi)$, and $(\theta = \frac{1}{2}\pi,\ b \le r \le a,\ 0 \le \phi < 2\pi)$. For this surface, use (12) to evaluate the left-hand side of (1) and show that it results in q.

1.3.3* A cylindrically symmetric charge configuration extends to infinity in the $\pm z$ directions and has the same cross-section in any constant z plane. Inside the radius b, the charge density has a parabolic dependence on radius while over the range $b < r < a$ outside that radius, the charge density is zero.

$$\rho = \begin{cases} \rho_o(r/b)^2; & r < b \\ 0; & b < r < a \end{cases} \tag{a}$$

There is no surface charge density at $r = b$.

(a) Use the axial symmetry and Gauss' integral law to show that $\mathbf{E}$ in the two regions is

$$\mathbf{E} = \begin{cases} (\rho_o r^3/4\epsilon_o b^2)\mathbf{i_r}; & r < b \\ (\rho_o b^2/4\epsilon_o r)\mathbf{i_r}; & b < r < a \end{cases} \tag{b}$$

(b) Outside a shell at $r = a$, $\mathbf{E} = 0$. Use (17) to show that the surface charge density at $r = a$ is

$$\sigma_s = -\rho_o b^2/4a \tag{c}$$

(c) Integrate this charge per unit area over the surface of the shell and show that the resulting charge per unit length on the shell is the negative of the charge per unit length inside.

(d) Show that, in Cartesian coordinates, $\mathbf{E}$ is

$$\mathbf{E} = \frac{\rho_o}{4\epsilon_o}\begin{cases} [x(x^2+y^2)/b^2]\mathbf{i_x} + [y(x^2+y^2)/b^2]\mathbf{i_y}; & r < b \\ b^2x(x^2+y^2)^{-1}\mathbf{i_x} + b^2y(x^2+y^2)^{-1}\mathbf{i_y}; & b < r < a \end{cases} \qquad (d)$$

Note that ($r = \sqrt{x^2+y^2}$, $\cos\phi = x/r$, $\sin\phi = y/r$, $\mathbf{i_r} = \mathbf{i_x}\cos\phi + \mathbf{i_y}\sin\phi$) and the result takes the form $\mathbf{E} = E_x(x,y)\mathbf{i_x} + E_y(x,y)\mathbf{i_y}$.

(e) Now, imagine that the circular cylinder of charge in the region $r < b$ is enclosed by a cylindrical surface of square cross-section with the z coordinate as its axis and unit length in the z direction. The walls of this surface are at $x = \pm c$, $y = \pm c$ and $z = 0$ and $z = 1$. (To be sure that the cylinder of the charge distribution is entirely within the surface, $b < r < a$, $b < c < a/\sqrt{2}$.) Show that the surface integral on the left in (1) is

$$\oint_S \epsilon_o\mathbf{E}\cdot d\mathbf{a} = \frac{\rho_o b^2}{4}\Big\{\int_{-c}^{c}\Big[\frac{c}{c^2+y^2} - \frac{(-c)}{c^2+y^2}\Big]dy + \int_{-c}^{c}\Big[\frac{c}{x^2+c^2} - \frac{(-c)}{x^2+c^2}\Big]dx\Big\} \qquad (e)$$

Without carrying out these integrations, what is the answer?

1.3.4 In a spherically symmetric configuration, the region $r < b$ has the uniform charge density ρ_b and is surrounded by a region $b < r < a$ having the uniform charge density ρ_a. At $r = b$ there is no surface charge density, while at $r = a$ there is that surface charge density that assures $\mathbf{E} = 0$ for $a < r$.

(a) Determine $\mathbf{E}$ in the two regions.

(b) What is the surface charge density at $r = a$?

(c) Now suppose that there is a surface charge density given at $r = b$ of $\sigma_s = \sigma_o$. Determine $\mathbf{E}$ in the two regions and σ_s at $r = a$.

1.3.5* The region between the plane parallel sheets of surface charge density shown in Fig. 1.3.8 is filled with a charge density $\rho = 2\rho_o z/s$, where ρ_o is a given constant. Again, assume that the electric field below the lower sheet is $E_o\mathbf{i_z}$ and show that between the sheets

$$E_z = E_o - \frac{\sigma_o}{\epsilon_o} + \frac{\rho_o}{\epsilon_o s}\big[z^2 - (s/2)^2\big] \qquad (a)$$

1.3.6 In a configuration much like that of Fig. 1.3.8, there are three rather than two sheets of charge. One, in the plane $z = 0$, has the given surface charge density σ_o. The second and third, respectively located at $z = s/2$ and

$z = -s/2$, have unknown charge densities σ_a and σ_b. The electric field outside the region $-\frac{1}{2}s < z < \frac{1}{2}s$ is zero, and $\sigma_a = 2\sigma_b$. Determine σ_a and σ_b.

1.3.7 Particles having charges of the same sign are constrained in their positions by a plastic tube which is tilted with respect to the horizontal by the angle α, as shown in Fig. P1.3.7. Given that the lower particle has charge Q_o and is fixed, while the upper one (which has charge Q and mass M) is free to move without friction, at what relative position, ξ, can the upper particle be in a state of static equilibrium?

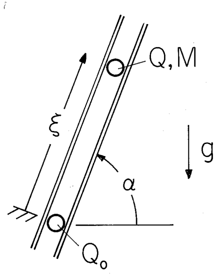

Fig. P1.3.7

1.4 Ampère's Integral Law

1.4.1* A static H field is produced by the cylindrically symmetric current density distribution $\mathbf{J} = J_o \exp(-r/a)\mathbf{i_z}$, where J_o and a are constants and r is the radial cylindrical coordinate. Use the integral form of Ampère's law to show that

$$H_\phi = \frac{J_o a^2}{r}\big[1 - e^{-r/a}\big(1 + \frac{r}{a}\big)\big] \qquad (a)$$

1.4.2* In polar coordinates, a uniform current density $J_o\mathbf{i_z}$ exists over the cross-section of a wire having radius b. This current is returned in the $-z$ direction as a uniform surface current at the radius $r = a > b$.

(a) Show that the surface current density at $r = a$ is

$$\mathbf{K} = -(J_o b^2/2a)\mathbf{i_z} \qquad (a)$$

(b) Use the integral form of Ampère's law to show that $\mathbf{H}$ in the regions $0 < r < b$ and $b < r < a$ is

$$\mathbf{H} = \begin{cases} (J_o r/2)\mathbf{i}_\phi; & r < b \\ (J_o b^2/2r)\mathbf{i}_\phi; & b < r < a \end{cases} \qquad (b)$$

(c) Use Ampère's continuity condition, (16), to show that $\mathbf{H} = 0$ for $r > a$.

(d) Show that in Cartesian coordinates, **H** is

$$\mathbf{H} = \frac{J_o}{2}\begin{cases} -y\mathbf{i_x} + x\mathbf{i_y}; & r < b \\ -b^2 y(x^2+y^2)^{-1}\mathbf{i_x} + b^2 x(x^2+y^2)^{-1}\mathbf{i_y}; & b < r < a \end{cases} \tag{c}$$

(e) Suppose that the inner cylinder is now enclosed by a contour C that encloses a square surface in a constant z plane with edges at $x = \pm c$ and $y = \pm c$ (so that C is in the region $b < r < a$, $b < c < a/\sqrt{2}$). Show that the contour integral on the left in (1) is

$$\begin{aligned}\oint_C \mathbf{H}\cdot d\mathbf{s} = &\int_{-c}^{c} \frac{J_o b^2}{2}\left(\frac{c}{c^2+y^2} - \frac{(-c)}{c^2+y^2}\right)dy \\ &+ \int_{-c}^{c} \frac{J_o b^2}{2}\left(\frac{c}{x^2+c^2} - \frac{(-c)}{x^2+c^2}\right)dx\end{aligned} \tag{d}$$

Without carrying out the integrations, use Ampère's integral law to deduce the result of evaluating (d).

1.4.3 In a configuration having axial symmetry about the z axis and extending to infinity in the $\pm z$ directions, a line current I flows in the $-z$ direction along the z axis. This current is returned uniformly in the $+z$ direction in the region $b < r < a$. There is no current density in the region $0 < r < b$ and there are no surface current densities.

(a) In terms of I, what is the current density in the region $b < r < a$?
(b) Use the symmetry of the configuration and the integral form of Ampère's law to deduce **H** in the regions $0 < r < b$ and $b < r < a$.
(c) Express **H** in each region in Cartesian coordinates.
(d) Now, consider the evaluation of the left-hand side of (1) for a contour C that encloses a square surface S having sides of length $2c$ and the z axis as a perpendicular. That is, C lies in a constant z plane and has sides $x = \pm c$ and $y = \pm c$ with $c < a/\sqrt{2}$). In Cartesian coordinates, set up the line integral on the left in (1). Without carrying out the integrations, what must the answer be?

1.4.4* In a configuration having axial symmetry about the z axis, a line current I flows in the $-z$ direction along the z axis. This current is returned at the radii a and b, where there are uniform surface current densities K_{za} and K_{zb}, respectively. The current density is zero in the regions $0 < r < b$, $b < r < a$ and $a < r$.

(a) Given that $K_{za} = 2K_{zb}$, show that $K_{za} = I/\pi(2a+b)$.
(b) Show that **H** is

$$\mathbf{H} = -\frac{I}{2\pi}\mathbf{i}_\phi \begin{cases} 1/r; & 0 < r < b \\ 2a/r(2a+b); & b < r < a \end{cases} \tag{a}$$

1.4.5 Uniform surface current densities $\mathbf{K} = \pm K_o \mathbf{i_y}$ are in the planes $z = \pm\frac{1}{2}s$, respectively. In the region $-\frac{1}{2}s < z < \frac{1}{2}s$, the current density is $\mathbf{J} = 2J_o z/s\mathbf{i_y}$. In the region $z < -\frac{1}{2}s$, $\mathbf{H} = 0$. Determine $\mathbf{H}$ for $-\frac{1}{2}s < z$.

1.5 Charge Conservation in Integral Form

1.5.1* In the region of space of interest, the charge density is uniform and a given function of time, $\rho = \rho_o(t)$. Given that the system has spherical symmetry, with r the distance from the center of symmetry, use the integral form of the law of charge conservation to show that the current density is

$$\mathbf{J} = -\frac{r}{3}\frac{d\rho_o}{dt}\mathbf{i_r} \qquad (a)$$

1.5.2 In the region $x > 0$, the charge density is known to be uniform and the given function of time $\rho = \rho_o(t)$. In the plane $x = 0$, the current density is zero. Given that it is x directed and only dependent on x and t, what is $\mathbf{J}$?

1.5.3* In the region $z > 0$, the current density $\mathbf{J} = 0$. In the region $z < 0$, $\mathbf{J} = J_o(x, y)\cos\omega t\mathbf{i_z}$, where J_o is a given function of (x, y). Given that when $t = 0$, the surface charge density $\sigma_s = 0$ over the plane $z = 0$, show that for $t > 0$, the surface charge density in the plane $z = 0$ is $\sigma_s(x, y, t) = [J_o(x, y)/\omega]\sin\omega t$.

1.5.4 In cylindrical coordinates, the current density $\mathbf{J} = 0$ for $r < R$, and $\mathbf{J} = J_o(\phi, z)\sin\omega t\mathbf{i_r}$ for $r > R$. The surface charge density on the surface at $r = R$ is $\sigma_s(\phi, z, t) = 0$ when $t = 0$. What is $\sigma_s(\phi, z, t)$ for $t > 0$?

1.6 Faraday's Integral Law

1.6.1* Consider the calculation of the circulation of $\mathbf{E}$, the left-hand side of (1), around a contour consisting of three segments enclosing a surface lying in the $x - y$ plane: from $(x, y) = (0, 0) \rightarrow (g, s)$ along the line $y = sx/g$; from $(x, y) = (g, s) \rightarrow (0, s)$ along $y = s$ and from $(x, y) = (0, s)$ to $(0, 0)$ along $x = 0$.

(a) Show that along the first leg, $d\mathbf{s} = [\mathbf{i_x} + (s/g)\mathbf{i_y}]dx$.
(b) Given that $\mathbf{E} = E_o\mathbf{i_y}$ where E_o is a given constant, show that the line integral along the first leg is sE_o and that the circulation around the closed contour is zero.

1.6.2 The situation is the same as in Prob. 1.6.1 except that the first segment of the closed contour is along the curve $y = s(x/g)^2$.

(a) Once again, show that for a uniform field $\mathbf{E} = E_o \mathbf{i_y}$, the circulation of $\mathbf{E}$ is zero.
(b) For $\mathbf{E} = E_o(x/g)\mathbf{i_y}$, what is the circulation around this contour?

1.6.3* The $\mathbf{E}$ field of a line charge density uniformly distributed along the z axis is given in cylindrical coordinates by (1.3.13).

(a) Show that in Cartesian coordinates, with $x = r\cos\phi$ and $y = r\sin\phi$,

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o}\left[\frac{x}{x^2+y^2}\mathbf{i_x} + \frac{y}{x^2+y^2}\mathbf{i_y}\right] \tag{a}$$

(b) For the contour shown in Fig. P1.6.3, show that

$$\oint_C \mathbf{E}\cdot d\mathbf{s} = \frac{\lambda_l}{2\pi\epsilon_o}\left[\int_k^g (1/x)dx + \int_0^h \frac{y}{g^2+y^2}dy - \int_k^g \frac{x}{x^2+h^2}dx - \int_0^h \frac{y}{k^2+y^2}dy\right] \tag{b}$$

and complete the integrations to prove that the circulation is zero.

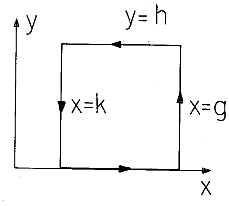

Fig. P1.6.3

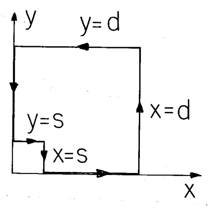

Fig. P1.6.4

1.6.4 A closed contour consisting of six segments is shown in Fig. P1.6.4. For the electric field intensity of Prob. 1.6.3, calculate the line integral of $\mathbf{E}\cdot d\mathbf{s}$ on each of these segments and show that the integral around the closed contour is zero.

1.6.5* The experiment in Fig. 1.6.4 is carried out with the coil positioned horizontally, as shown in Fig. 1.7.2. The left edge of the coil is directly below the wire, at a distance d, while the right edge is at the radial distance R from the wire, as shown. The area element $d\mathbf{a}$ is y directed (the vertical direction).

(a) Show that, in Cartesian coordinates, the magnetic field intensity due to the current i is

$$\mathbf{H} = \frac{i}{2\pi}\left(\frac{-\mathbf{i_x} y}{x^2+y^2} + \frac{\mathbf{i_y} x}{x^2+y^2}\right) \qquad (a)$$

(b) Use this field to show that the magnetic flux linking the coil is as given by (1.7.5).
(c) What is the circulation of $\mathbf{E}$ around the contour representing the coil?
(d) Given that the coil has N turns, what is the EMF measured at its terminals?

1.6.6 The magnetic field intensity is given to be $\mathbf{H} = H_o(t)(\mathbf{i_x}+\mathbf{i_y})$, where $H_o(t)$ is a given function of time. What is the circulation of $\mathbf{E}$ around the contour shown in Fig. P1.6.6?

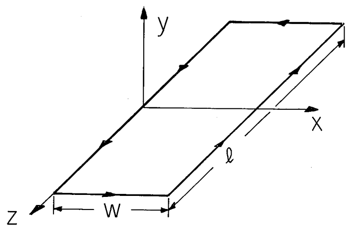

Fig. P1.6.6

1.6.7* In the plane $y = 0$, there is a uniform surface charge density $\sigma_s = \sigma_o$. In the region $y < 0$, $\mathbf{E} = E_1\mathbf{i_x} + E_2\mathbf{i_y}$ where E_1 and E_2 are given constants. Use the continuity conditions of Gauss and Faraday, (1.3.17) and (12), to show that just above the plane $y = 0$, where $y = 0^+$, the electric field intensity is $\mathbf{E} = E_1\mathbf{i_x} + [E_2 + (\sigma_o/\epsilon_o)]\mathbf{i_y}$.

1.6.8 Inside a circular cylindrical region having radius $r = R$, the electric field intensity is $\mathbf{E} = E_o\mathbf{i_y}$, where E_o is a given constant. There is a surface charge density $\sigma_o \cos\phi$ on the surface at $r = R$ (the polar coordinate ϕ is measured relative to the x axis). What is $\mathbf{E}$ just outside the surface, where $r = R^+$?

1.7 Integral Magnetic Flux Continuity Law

1.7.1* A region is filled by a uniform magnetic field intensity $H_o(t)\mathbf{i_z}$.

(a) Show that in spherical coordinates (defined in Fig. A.1.3 of Appendix 1), $\mathbf{H} = H_o(t)(\mathbf{i_r}\cos\theta - \mathbf{i_\theta}\sin\theta)$.
(b) A circular contour lies in the $z = 0$ plane and is at $r = R$. Using the enclosed surface in the plane $z = 0$ as the surface S, show that the circulation of $\mathbf{E}$ in the ϕ direction around C is $-\pi R^2\mu_o dH_o/dt$.

(c) Now compute the same circulation using as a surface S enclosed by C the hemispherical surface at $r = R$, $0 \le \theta < \frac{1}{2}\pi$.

1.7.2 With $H_o(t)$ a given function of time and d a given constant, three distributions of $\mathbf{H}$ are proposed.

$$\mathbf{H} = H_o(t)\mathbf{i_y} \qquad (a)$$

$$\mathbf{H} = H_o(t)(x/d)\mathbf{i_x} \qquad (b)$$

$$\mathbf{H} = H_o(t)(y/d)\mathbf{i_x} \qquad (c)$$

Which one of these will not satisfy (1) for a surface S as shown in Fig. 1.5.3?

1.7.3* In the plane $y = 0$, there is a given surface current density $\mathbf{K} = K_o\mathbf{i_x}$. In the region $y < 0$, $\mathbf{H} = H_1\mathbf{i_y} + H_2\mathbf{i_z}$. Use the continuity conditions of (1.4.16) and (6) to show that just above the current sheet, where $y = 0^+$, $\mathbf{H} = (H_1 - K_o)\mathbf{i_y} + H_z\mathbf{i_z}$.

1.7.4 In the circular cylindrical surface $r = R$, there is a surface current density $\mathbf{K} = K_o\mathbf{i_z}$. Just inside this surface, where $r = R$, $\mathbf{H} = H_1\mathbf{i_r}$. What is $\mathbf{H}$ just outside the surface, where $r = R^+$?